Error-Free Communication Over State-Dependent Channels with Variable-Length Feedback

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Joint work with

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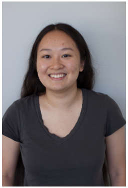

Carol Wang (NUS)

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  - The state process is i.i.d.

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- Notation:
  - $\mathcal{M}$ – the set of messages
  - $\mathcal{X}$ – channel input alphabet
  - $\mathcal{Y}$ – channel output alphabet
  - $\mathcal{S}$ – the set of channel states
  - M, X, Y, S – r.v.s corresponding to the selected message, channel input letter, channel output letter, and channel state

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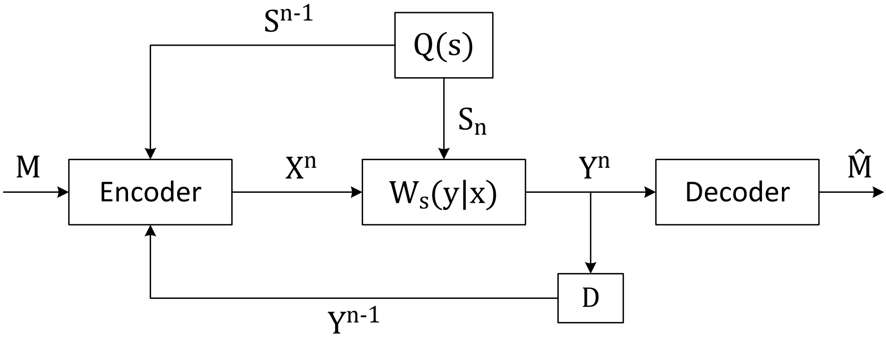

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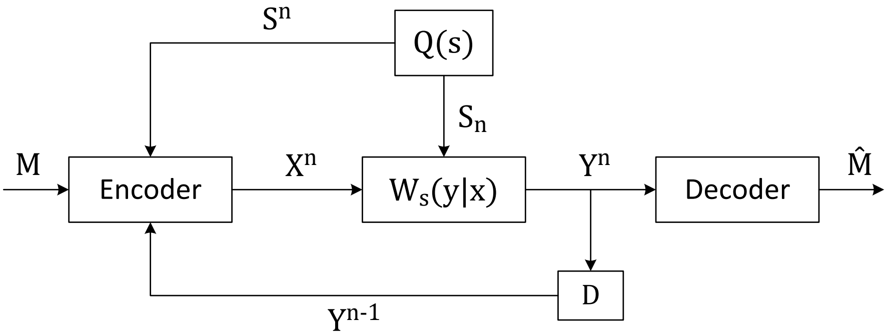
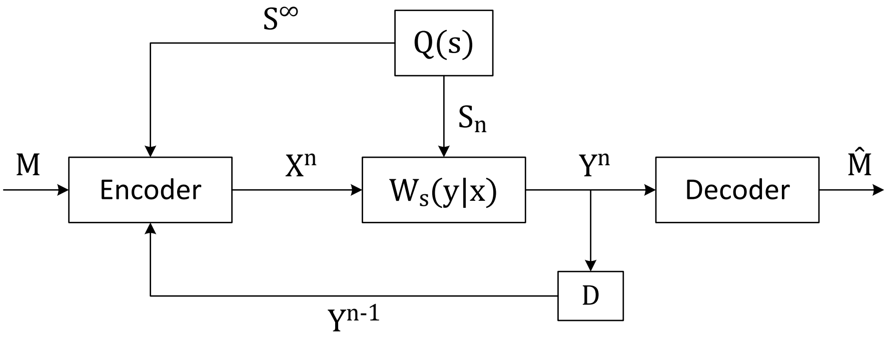


- We analyze various cases of state information availability: non-causal state info at the transmitter (Gel'fand-Pinsker)



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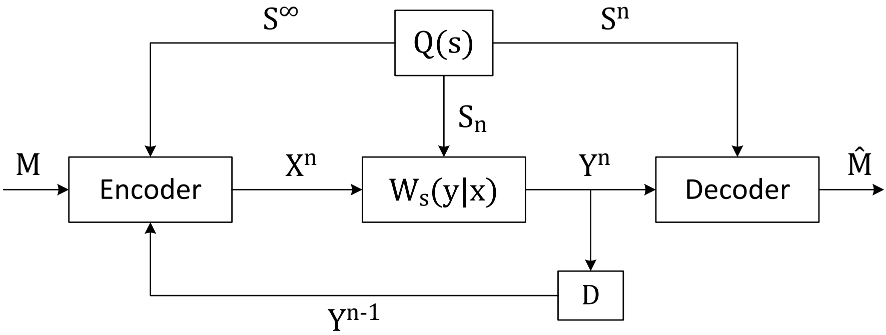
- We analyze various cases of state information availability: state info at both the transmitter and the receiver



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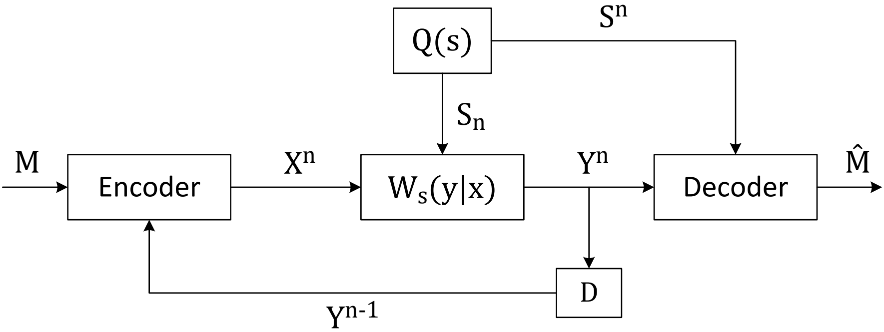

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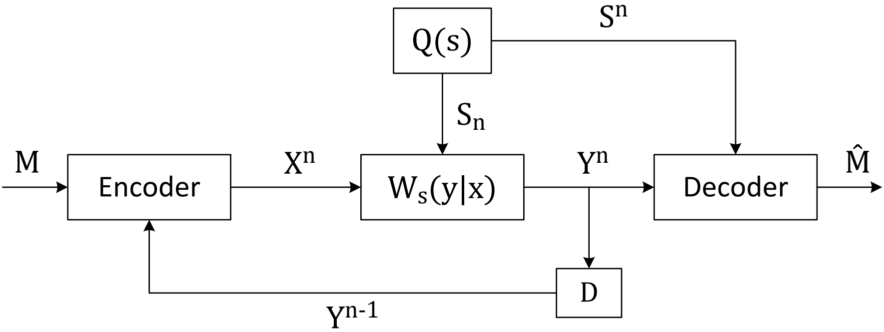

Variable-Length Coding Schemes

- SD-DMC with causal state information at both Enc and Dec

Definition

An $(\ell, |\mathcal{M}|, \epsilon)$ variable-length feedback (VLF) code for the message set $\mathcal{M}$ is defined by:

1) A sequence of encoders $f_n : \mathcal{M} \times \mathcal{Y}^{n-1} \times \mathcal{S}^n \to \mathcal{X}$, defining

$$X_n = f_n(M, Y^{n-1}, S^n);$$

2) A sequence of decoders $g_n : \mathcal{Y}^n \times \mathcal{S}^n \to \mathcal{M}$, defining the decoder's estimates of the transmitted message, $g_n(Y^n, S^n)$;

3) A integer-valued random variable τ (stopping time of $\{\sigma(Y^n, S^n)\}_{n=0}^{\infty}$) representing the code length and satisfying

$$\mathbb{E}[\tau] \leq \ell.$$

Variable-Length Coding Schemes

Definition (cont.)

Decoder's final decision is computed at time τ,

$$\widehat{M} = g_\tau(Y^\tau, S^\tau),$$

and it must satisfy

$$\mathbb{P}\left[\widehat{M} \neq M\right] \leq \epsilon.$$

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When $\epsilon = 0$, such a code is called zero-error VLF code.

If there exists a constant $b < \infty$ such that $\tau \leq b$, such a code is called bounded-length feedback code, and if $\tau = b = \ell$, it is called fixed-length feedback code.

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For the other cases of state information availability –
replace S^n above by S^0, S^{n-1}, or S^∞ accordingly.

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Code rate: $\frac{1}{\ell} \log |\mathcal{M}|$.

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$$C = \sup\left\{\frac{1}{\ell}\log|\mathcal{M}| : \lim_{\ell\to\infty}\mathbb{P}\left[\widehat{M}\neq M\right] = 0\right\}$$

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- For each of the above capacities, one must specify variable-length, bounded-length, of fixed-length coding schemes.

Main Results

- We determine the zero-error VLF capacity of SD-DMCs by:
  - finding necessary and sufficient conditions for positivity of the zero-error VLF capacity
  - proving that the zero-error VLF capacity, whenever positive, equals the conventional, vanishing-error capacity of the same channel
- We obtain the corresponding results for the bounded-length coding schemes as well
- arXiv:1712.07756

Related Work

- Zero-error capacity of channels with feedback:
  - Shannon'56: DMCs, fixed-length codes
  - Burnashev'76: DMCs, variable-length codes
  - Han–Sato'91: DMCs, bounded-length codes
  - Massey'07: DMCs, variable-length codes (different proofs, noisy feedback, incomplete feedback, etc.)
  - Bracher–Lapidoth'18: SD-DMCs with channel state info the the transmitter, fixed-length codes

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Theorem

The zero-error VLF capacity of the Gel'fand-Pinsker channel is positive if and only if

$$\exists y \in \mathcal{Y} \quad \forall s \in \mathcal{S} \quad \exists x_s \in \mathcal{X} \quad W(y|x_s, s) = 0. \tag{1}$$

- If (1) holds, y is said to be a disprover for input x_s in the state s (Massey's terminology)

Zero-Error Capacity: Variable-Length Codes

Proof $(\Leftarrow)$**:**

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- Also, let $x'_s \neq x_s$ be an input letter satisfying

$$W(y|x'_s, s) > 0$$

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Proof $(\Leftarrow)$:

- If the states realized in the first two time slots are s_1, s_2, the transmitter sends x_{s_1}, x'_{s_2} for 0 and x'_{s_1}, x_{s_2} for 1
  Note that the transmitter knows the states (Gel'fand-Pinsker)

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  - if the letters obtained at the output are $y, \neg y$, then 1 must have been transmitted;
  - if the letters obtained at the output are $\neg y, \neg y$, the procedure is repeated in the next two slots, and so on.
- In a finite expected number of channel uses the receiver will recover the transmitted bit.

Zero-Error Capacity: Variable-Length Codes

Proof $(\Rightarrow)$**:**

- If (1) doesn't hold, then for every output letter y there exists a state s_y such that $W(y|x, s_y) > 0$ for all input letters x

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- Therefore, zero-error communication in a finite average number of channel uses is impossible. ■

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Theorem

The zero-error VLF capacity of an SD-DMC, whenever positive, equals the vanishing-error capacity of the same channel.

The statement holds in all cases of state-information availability

- none
- strictly causal
- causal
- non-causal

and at the

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Proof: (Massey's coding scheme)

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  - ACK means that the decoding was correct and that a new codeword is about to be transmitted
  - NACK means that the decoding was not correct and that the same codeword is about to be retransmitted

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- In a finite expected number of channel uses the receiver will correctly decode the transmitted codeword

Zero-Error Capacity: Variable-Length Codes

Proof: (Massey's coding scheme)

- In a finite expected number of channel uses the receiver will correctly decode the transmitted codeword
- The rate of the whole scheme is also $\approx C$ because retransmissions happen with probability ϵ, and this can be made arbitrarily small ■

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The conditions for positivity of the zero-error feedback capacity under bounded-length coding are the same as under fixed-length coding.

- For example, for the Gel'fand-Pinsker channel, this condition is (Bracher–Lapidoth'18):

$$\forall s, s' \in \mathcal{S} \quad \exists x, x' \in \mathcal{X} \quad \forall y \in \mathcal{Y} \quad W(y|x,s)W(y|x',s') = 0.$$

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- The case where state information is given only to the receiver seems to be quite subtle
- It is not clear how to define the code length for variable-length codes, i.e., the stopping time of the transmission
  - The decoder makes a decision based on the outputs Y^n and states S^n
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  - The encoder, not knowing the states, is not able to exactly simulate the decoding process and to determine the moment when the decision has been made
- For this case, we obtain two sufficient conditions for positivity of the zero-error VLF capacity

Further Work

- Incomplete/noisy/coded feedback
- Zero-error VLF capacity for the case when state information is given only to the decoder
- Other models: non i.i.d. channel states, multi-user channels...
- etc.