On Third-Order Asymptotics for DMCs

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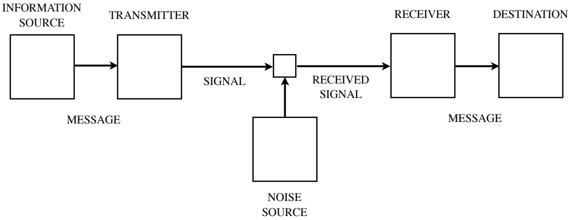
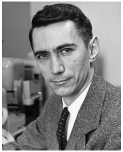
Acknowledgements

This is joint work with Marco Tomamichel

Centre for Quantum Technologies
National University of Singapore

Transmission of Information





Shannon's Figure 1

- Information theory $\equiv$ Finding fundamental limits for reliable information transmission

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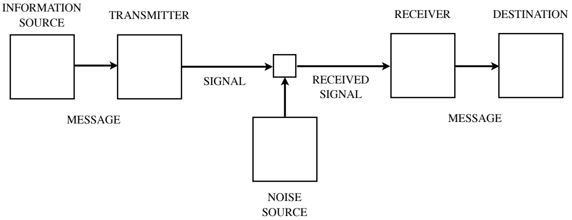
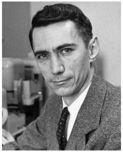
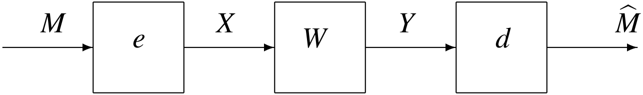




Shannon's Figure 1

- Information theory $\equiv$ Finding fundamental limits for reliable information transmission
- Channel coding: Concerned with the maximum rate of communication in bits/channel use

Channel Coding (One-Shot)



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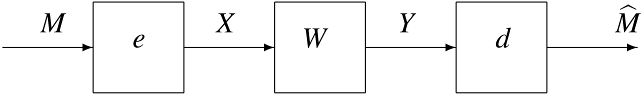

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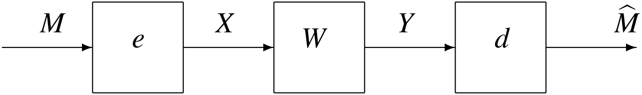

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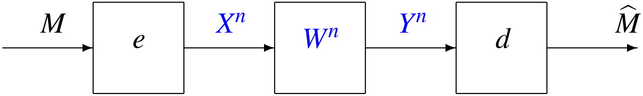
- ε-Error Capacity is

$$M^*(W,\varepsilon) := \sup\left\{m \in \mathbb{N} \,\middle|\, \exists\, \mathcal{C} \text{ s.t. } m = |\mathcal{M}|,\, p_{\mathrm{err}}(\mathcal{C}) \le \varepsilon\right\}$$

Channel Coding (n-Shot)



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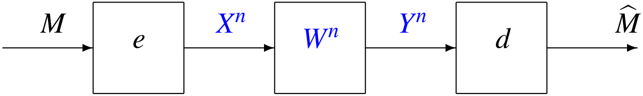

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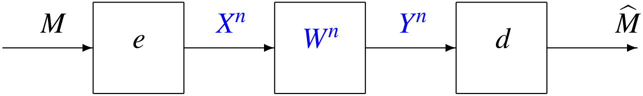

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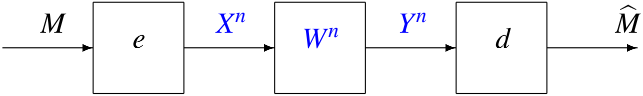

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- Blocklength n, ε-Error Capacity is

$$M^*(W^n, \varepsilon)$$

Main Contribution

- Upper bound $\log M^*(W^n, \varepsilon)$ for n large (converse)

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Theorem (Tomamichel-Tan (2013))

For all DMCs with positive ε-dispersion V_ε,

$$\log M^*(W^n, \varepsilon) \le nC - \sqrt{nV_\varepsilon}\mathrm{Q}^{-1}(\varepsilon) + \frac{1}{2}\log n + O(1)$$

where $\mathrm{Q}(a) := \int_a^{+\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}x^2\right)\,\mathrm{d}x$

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- The $\frac{1}{2}\log n$ term is our main contribution

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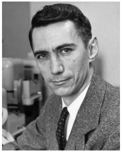
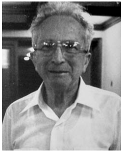
- Requires new converse techniques

Outline

Background: Shannon's Channel Coding Theorem

- Shannon's noisy channel coding theorem and
- Wolfowitz's strong converse state that





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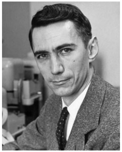

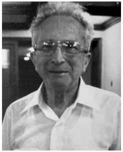

Theorem (Shannon (1949), Wolfowitz (1959))

$$\lim_{n\to\infty} \frac{1}{n} \log M^*(W^n, \varepsilon) = C, \qquad \forall\, \varepsilon \in (0,1)$$

where C is the channel capacity defined as

$$C = C(W) = \max_P I(P, W)$$

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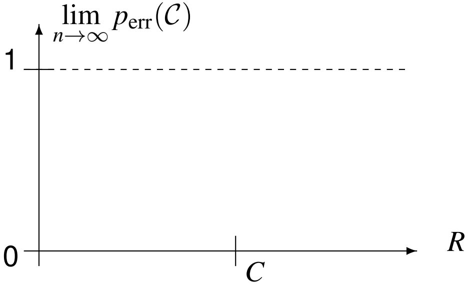

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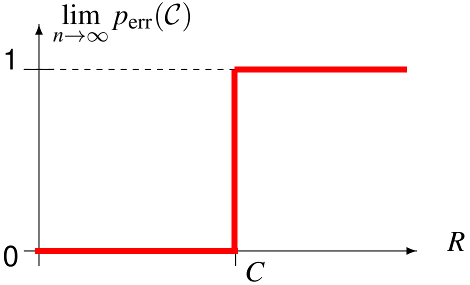

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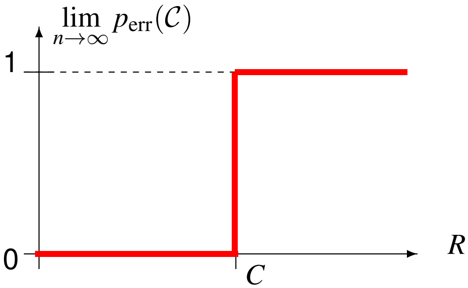

- Phase transition at capacity

Background: ε-Dispersion

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- Assume capacity-achieving input distribution (CAID) P^* is unique
- The ε-dispersion is an operational quantity that is equal to

$$V_\varepsilon = V(P^*, W) = \mathbb{E}_{P^*}\left[\mathrm{Var}_{W(\cdot|X)}\left(\log \frac{W(\cdot|X)}{Q^*(\cdot)} \,\Big|\, X\right)\right]$$

where $(X, Y) \sim P^* \times W$ and $Q^*(y) = \sum_x P^*(x)W(y|x)$

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- Since CAID is unique, $V_\varepsilon = V$

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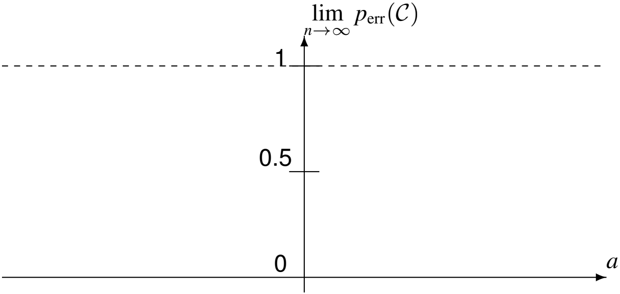

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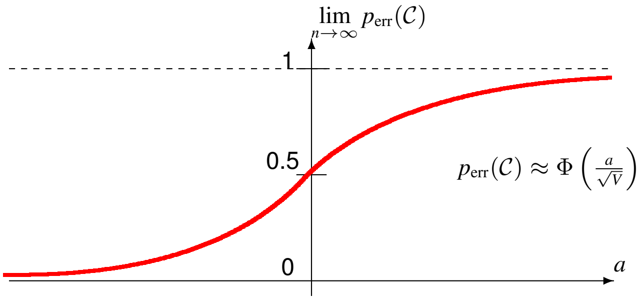

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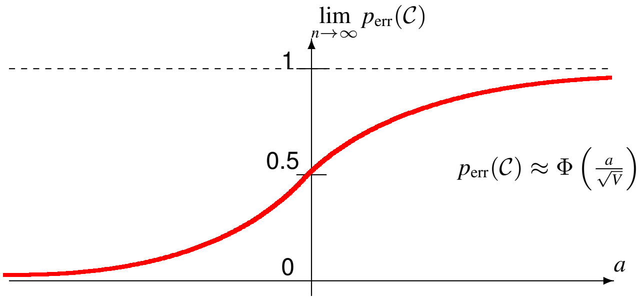

Here, we have fixed a, the second-order coding rate [Hayashi (2009)]

Background: ε-Dispersion

Theorem (Strassen (1964), Hayashi (2009), Polyanskiy-Poor-Verdú (2010))

For every $\varepsilon \in (0,1)$, and if $V_\varepsilon > 0$, we have

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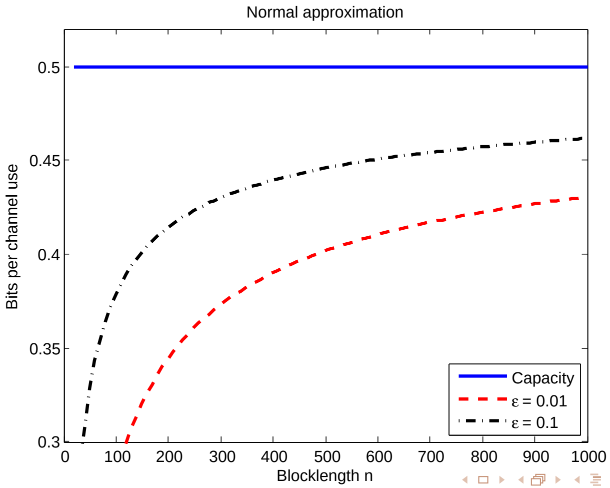
- PPV showed that the normal approximation

$$\log M^*(W^n, \varepsilon) \approx nC - \sqrt{nV}\mathrm{Q}^{-1}(\varepsilon)$$

  is very accurate even at moderate blocklengths of ≈ 100

Background: ε-Dispersion for the BSC

For a BSC with crossover probability $p = 0.11$, the normal approximation yields:



Related Work: Third-Order Term

- Recall that we are interested in quantifying the third-order term ρ_n

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- Motivation 2: Because we're information theorists

Wir müssen wissen – wir werden wissen (David Hilbert)

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$$\rho_n = \frac{1}{2}\log n + O(1)$$

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- Our converse technique can be applied to the AWGN channel

Related Work: Achievability for Third-Order Term

Proposition (Polyanskiy (2010))

Assume that all elements of $\{W(y|x) : x \in \mathcal{X}, y \in \mathcal{Y}\}$ *are positive and* $C > 0$*. Then,*

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- BEC doesn't satisfy assumptions
- We will not try to improve on it

Related Work: Converse for Third-Order Term

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- We dispense of this symmetry assumption

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If W is a DMC with positive ε-dispersion,

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- This is where the dependence on $|\mathcal{X}|$ comes in

Main Result: Tight Third-Order Term

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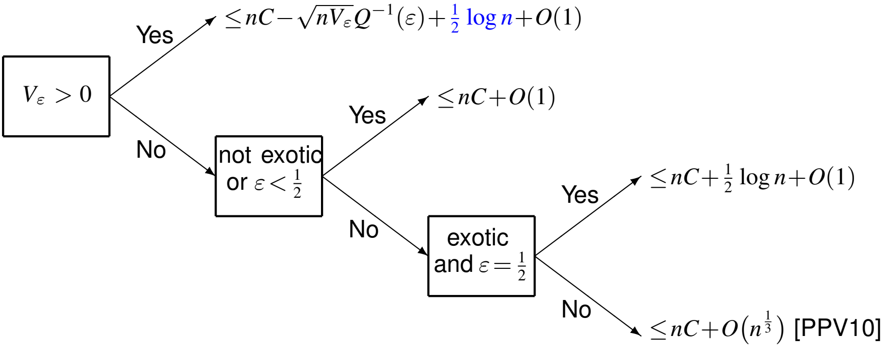
All cases are covered



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"Information Spectrum Methods in Information Theory"
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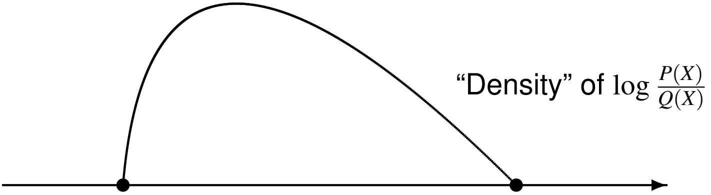

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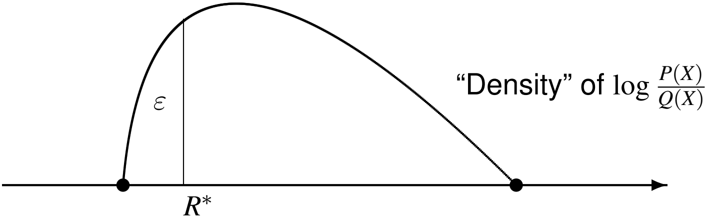

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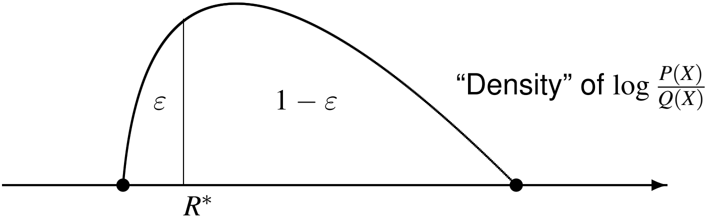

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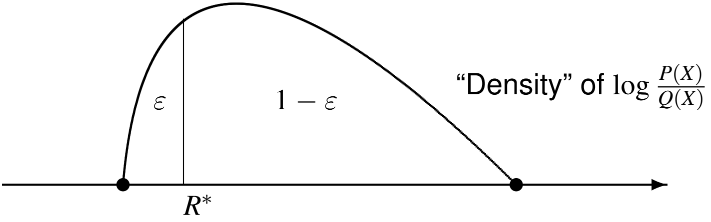

If X^n is i.i.d. P, the central limit theorem yields

$$D_s^\varepsilon(P^n\|Q^n) \approx nD(P\|Q) - \sqrt{nV(P\|Q)}\mathrm{Q}^{-1}(\varepsilon)$$

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- Since all $\mathbf{x}$ within a type class result in the same $D_s^{\varepsilon+\delta}$ (if $Q^{(n)}$ is permutation invariant), it's really a $\max$ over types $P_{\mathbf{x}} \in \mathcal{P}_n(\mathcal{X})$

Proof Technique: Choice of Output Distribution

$$\log M^*(W^n, \varepsilon) \le \max_{\mathbf{x}\in\mathcal{X}^n} D_s^{\varepsilon+\delta}(W^n(\cdot|\mathbf{x})\|Q^{(n)}) + \log\frac{1}{\delta}, \qquad \forall\, Q^{(n)} \in \mathcal{P}(\mathcal{Y}^n)$$

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- Second term: Mixture over output distributions induced by input types [Hayashi (2009)]

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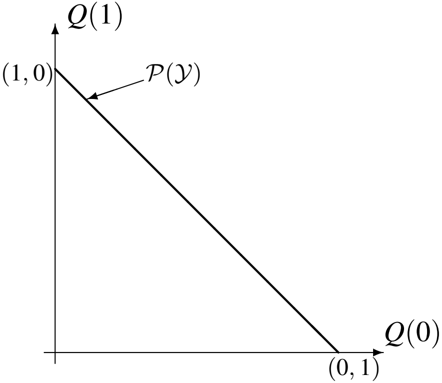

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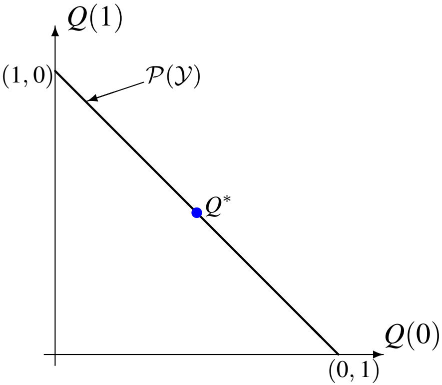

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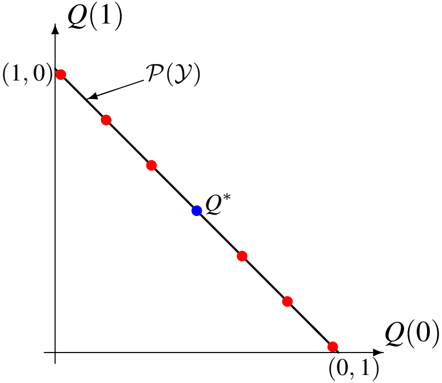

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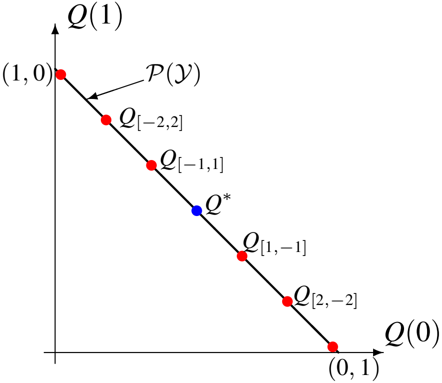

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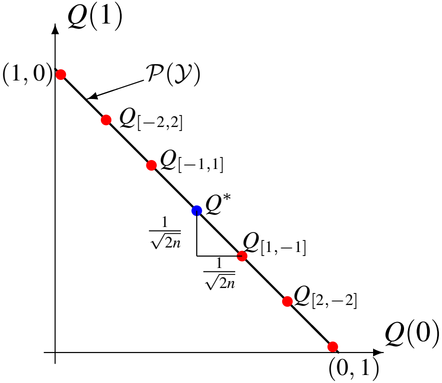

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$$I(P_{\mathbf{x}}, W) \le C' < C$$

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- This result has been used to refine the sphere-packing bound [Altug-Wagner (2012)]