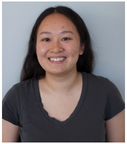
On the Maximum Size of Block Codes Subject to a Distance Criterion

Vincent Y. F. Tan

National University of Singapore (NUS)



Ling-Hua Chang — Yuan Ze Univ.

Po-Ning Chen — NCTU

Carol Wang — NUS

Yunghsiang Han — Dongguan Univ. of Tech.

ITCom Workshop (Jan 2019)

Error-correcting codes

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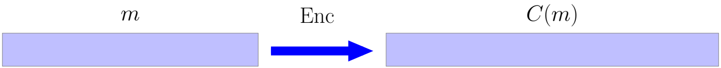

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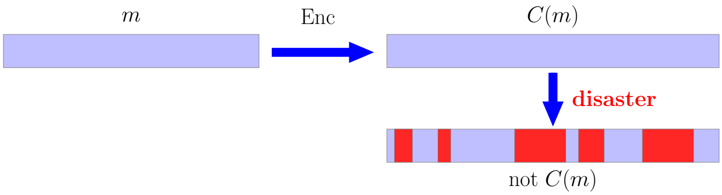

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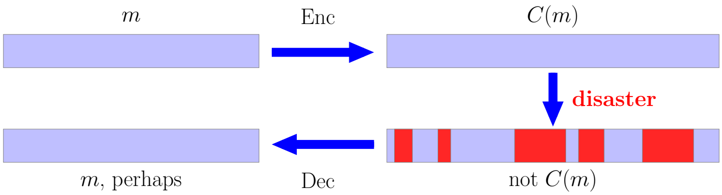

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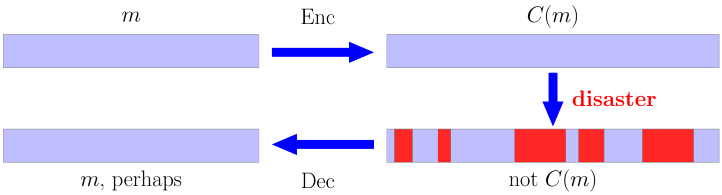

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Set of codewords is a code $\mathcal{C}$.

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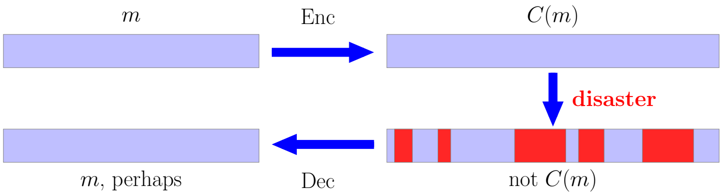

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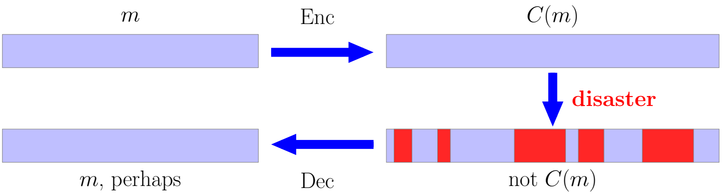

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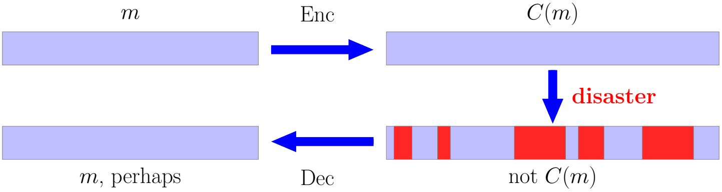

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- Distance : error-correction potential

Distance and errors

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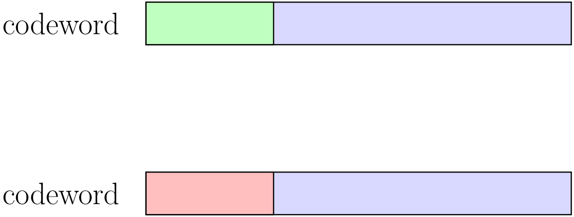

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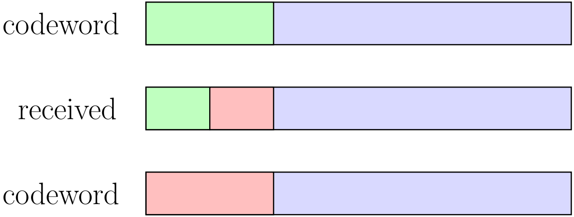

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(deletion distance, rank-metric, etc)

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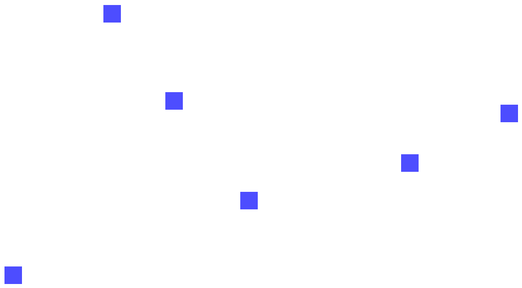

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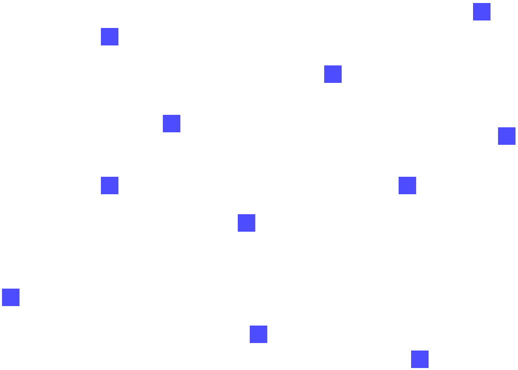

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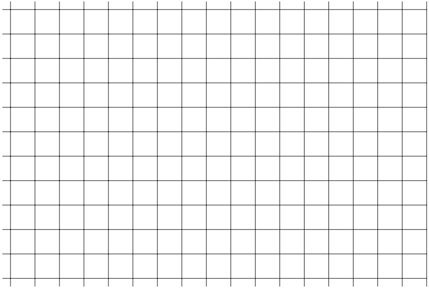

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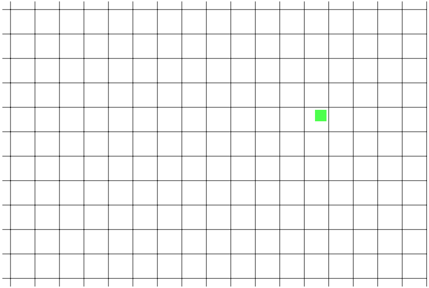

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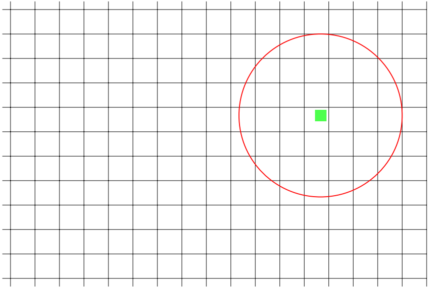

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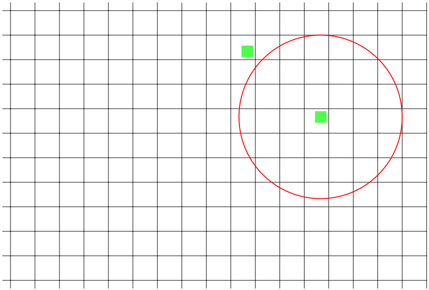

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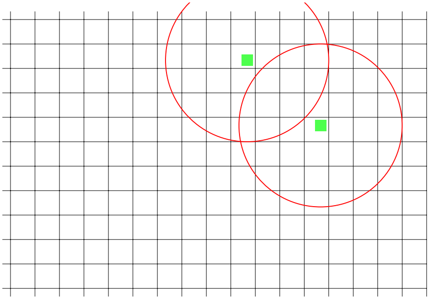

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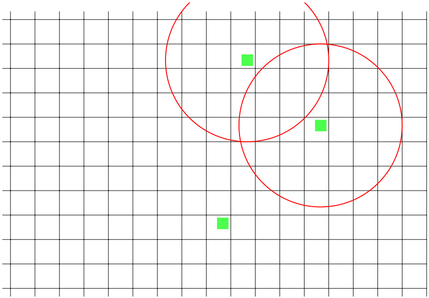

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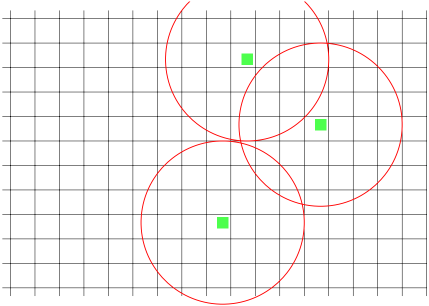

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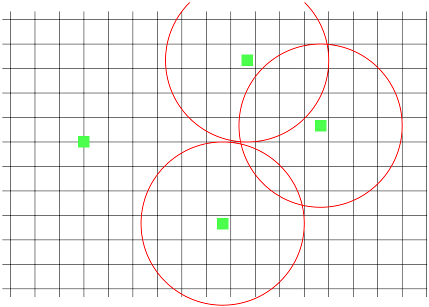

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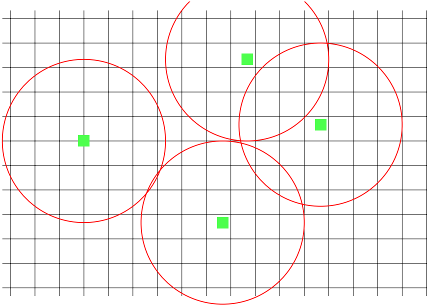

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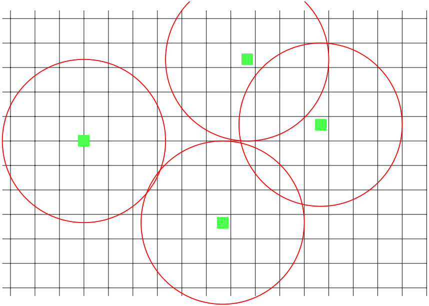

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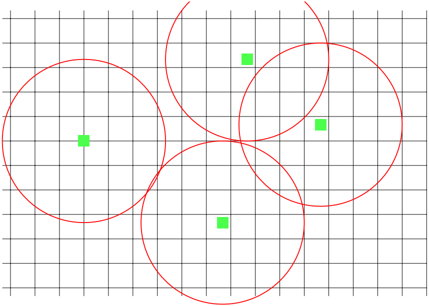

Each circle has $\approx 2^{H(\delta)n}$ vectors, so final code size is $2^n / 2^{H(\delta)n}$.

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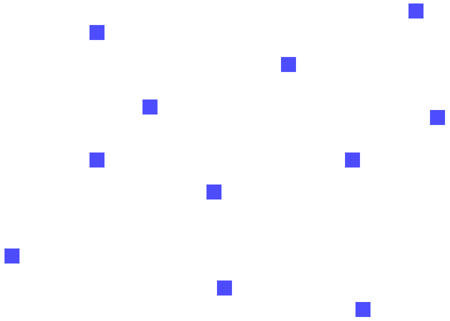

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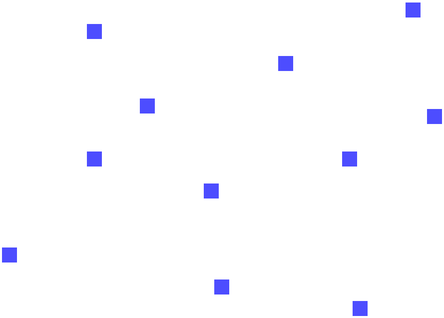

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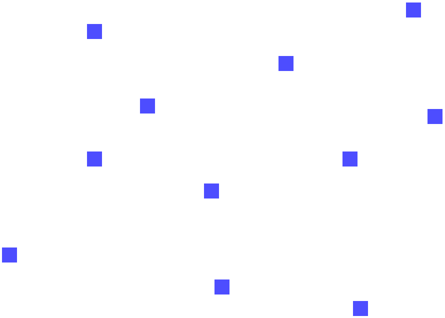

Works for rate $R \approx 1 - H(\delta)$ (proof on next slide).

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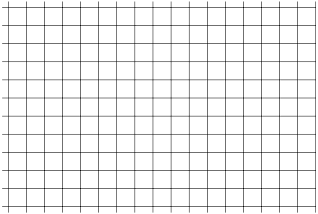

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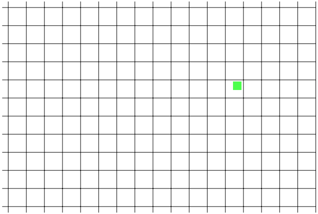

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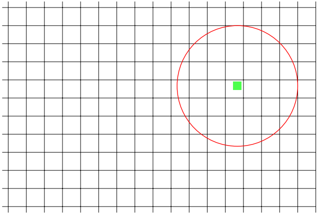

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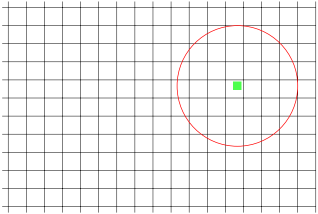

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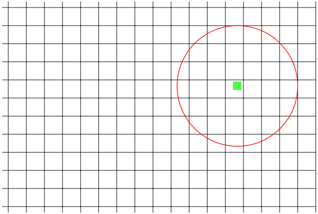

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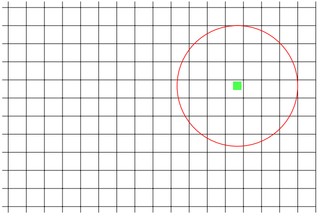

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Number of "bad" pairs $(\mathbf{x}, \mathbf{y})$ is

$$\approx 2^{2Rn} \cdot \frac{2^{H(\delta)n}}{2^n} = 2^{(R-\epsilon)n}.$$

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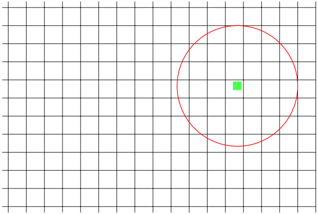

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Remove one element from each bad pair.

Distance is now δ, and rate is still $\approx R$.

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To mimic the GV proof, need to understand collision probability.

When are two random codewords at distance $< d$?

In other words. . .

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Moral: For various $\mathbf{X}$, want to understand collision probability (distance spectrum):

$$F_{\mathbf{X}}(d) := \Pr\left[\mu(\mathbf{X}, \hat{\mathbf{X}}) < d\right],$$

where $\hat{\mathbf{X}}$ is an independent copy of $\mathbf{X}$.

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Key points:

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- No asymptotics!
- Exact formula for basically any distance measure.

Exact distance spectrum formula

So, if $\mathbf{X}$ is uniform over $\mathcal{C}$, then

$$|\mathcal{C}| = \frac{1}{F_{\mathbf{X}}(d)}.$$

In fact, this is tight.

Theorem (Main theorem)

Let $M^*(d)$ be the optimal size of a distance d code. Then

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- Holds for arbitrary (non-discrete) alphabets.

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- Turns question about codes into one about distributions.
- Allows us to use optimization techniques for distributions.
- New bounds on the second-order asymptotics.
- Best distribution is uniform over optimal code, but any distribution gives a lower bound.

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For a fixed random vector $\mathbf{X}$, want to show:

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2. If $M > M^*(d)$, can reduce to first case.

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So, for small support, uniform is best.

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If we iterate this until the support has size $M^*(d)$, then

$$F_{\mathbf{X}}(d) \geq F_{\mathbf{X}'}(d) \geq F_{\mathbf{X}''}(d) \geq \cdots \geq \frac{1}{M^*(d)}.$$

Large support cont.

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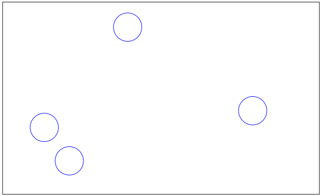

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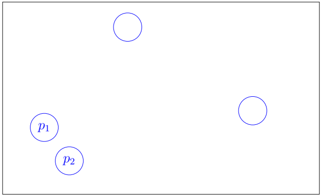

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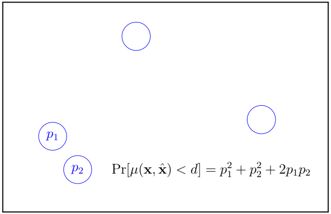

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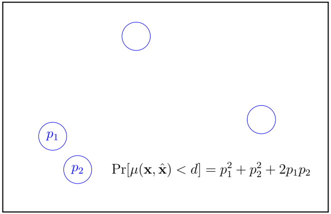

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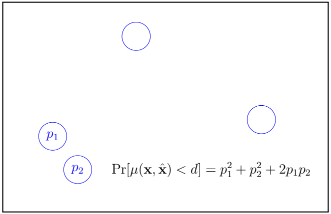

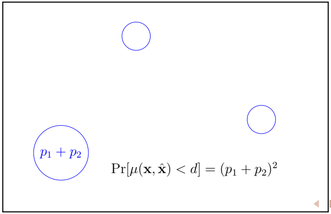

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Keeps distance spectrum (collision probability) $F_{\mathbf{X}}(d)$ small.

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(Upper bound via uniform distribution.)

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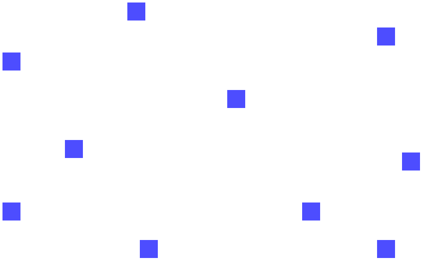

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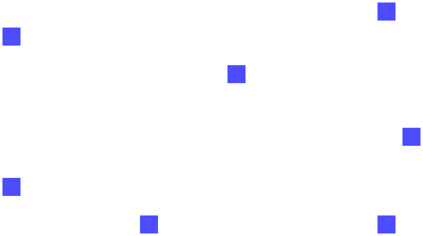

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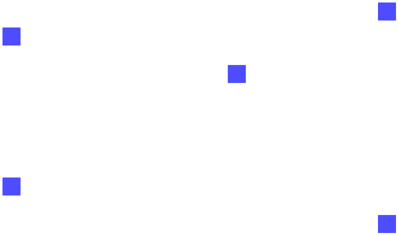

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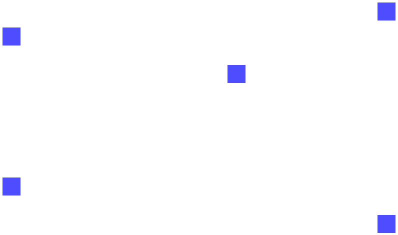

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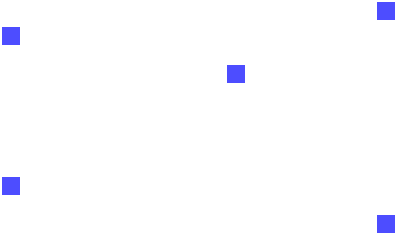
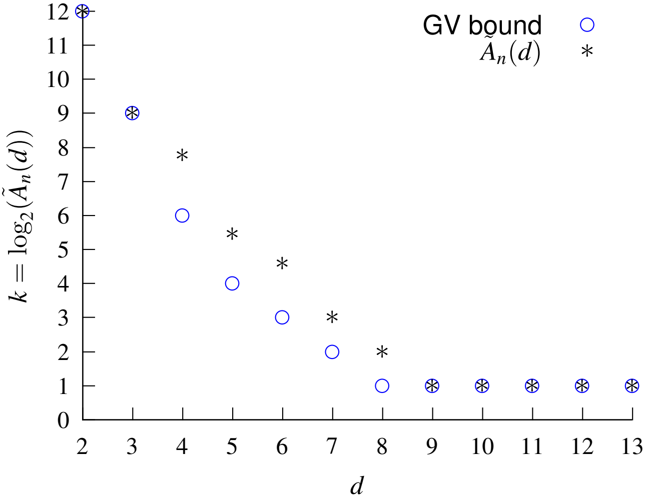


Can be thought of as a different way to implement GV greedy construction. Seems to work well in simulations.

An Algorithmic Construction $(n = 13)$



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- We now generalize to the case in which $|\mathcal{X}| = \infty$ (even uncountable)
- Idea: Greedy selection of codewords $\{\mathbf{u}_i\}_{i=1}^k$ given a fixed random vector/distribution $\mathbf{X} \sim P_{\mathbf{X}}$.

Non-Discrete Code Alphabets: Illustration

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$$\mathbf{u}_2 = \arg\min_{\mathbf{u}_2} \Pr\left[\mathbf{X} \in \mathcal{B}_d(\mathbf{u}_2) \setminus \mathcal{B}_d(\mathbf{u}_1)\right]$$

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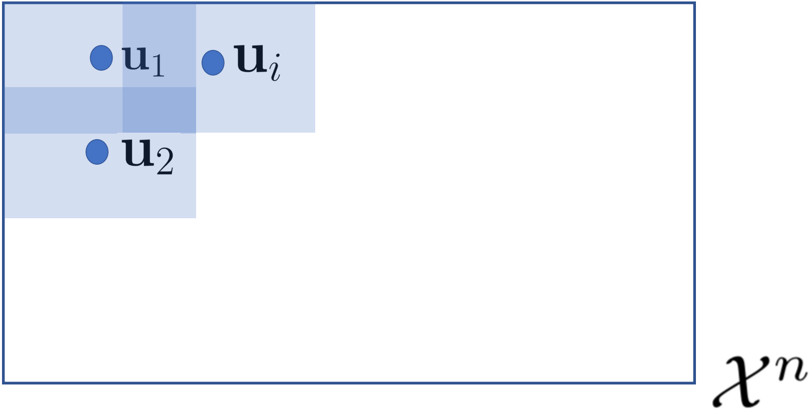
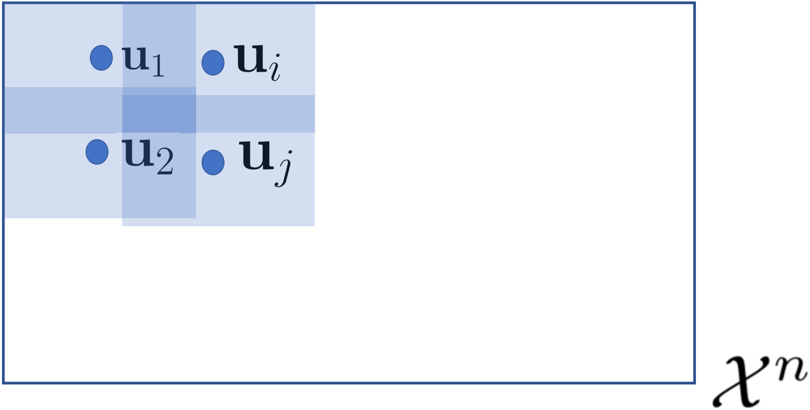


$$\mathbf{u}_i = \arg\min_{\mathbf{u}_i} \Pr\left[\mathbf{X} \in \mathcal{B}_d(\mathbf{u}_i) \setminus \cup_{j=1}^{i-1} \mathcal{B}_d(\mathbf{u}_j)\right]$$



Choose more centers $\mathbf{u}_j$'s not in preceding balls.

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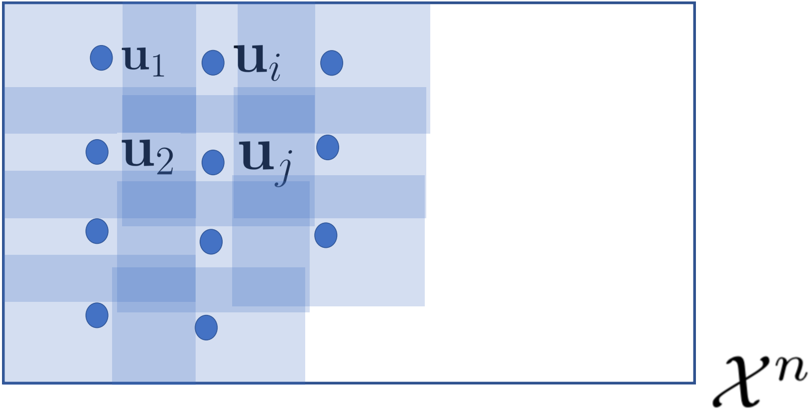

And more balls...

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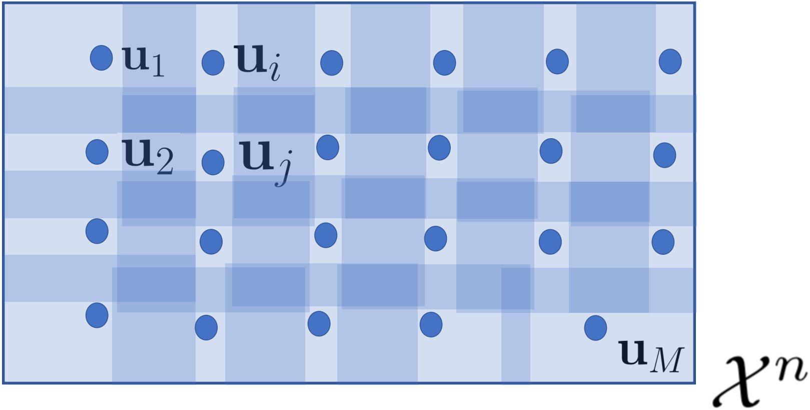

Until you run out of space!

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Let $\mathcal{D}_i := \mathcal{B}_d(\mathbf{u}_i) \setminus \cup_{j=1}^{i-1} \mathcal{B}_d(\mathbf{u}_j)$ and note that $\{\mathcal{D}_i\}$ forms a partition of $\mathcal{X}^n$.

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$$\Pr[\mu(\mathbf{X}, \hat{\mathbf{X}}) < d] = \sum_{j=1}^{M} \int_{\mathbf{x} \in \mathcal{D}_j} \left(\int_{\hat{\mathbf{x}} \in \mathcal{B}_d(\mathbf{x})} dP_{\mathbf{X}}(\hat{\mathbf{x}}) \right) dP_{\mathbf{X}}(\mathbf{x}) \quad \because \mathbf{X} \perp\!\!\!\perp \hat{\mathbf{X}}$$

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\end{aligned}$$

Non-Discrete Code Alphabets: Achievability Proof

The code $\mathcal{C} = \{\mathbf{u}_i : i = 1, \dots, M\}$ formed is a distance-d code and

$$p_j := \Pr\left[\mathbf{X} \in \mathcal{B}_d(\mathbf{u}_i) \setminus \cup_{j=1}^{i-1} \mathcal{B}_d(\mathbf{u}_j)\right], \quad \text{satisfies} \quad \sum_{j=1}^{M} p_j = 1.$$

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&\ge \sum_{j=1}^{M} p_j^2 \ge \frac{1}{M} \ge \frac{1}{M^*(d)} \quad \because \text{Cauchy-Schwarz } \& \ M \le M^*(d)
\end{aligned}$$

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Corollary (Refined GV bound)

For the Hamming distance, the optimal code rate for distance δn is

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Jiang and Vardy (2004) showed that the “second-order term” $\geq \frac{\log n}{n}$.

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Corollary (Upper Bound on Rate)

For any arbitrary bounded distance measure, the optimal code rate for distance δn is

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$$I_{X^n}(a) := \sup_{\theta} \left\{ a\theta - \varphi_{X^n}(\theta) \right\}, \quad \text{and} \quad \varphi_X(\theta) := \log \mathbb{E}\left[e^{\theta \mu(X, \hat{X})} \right].$$

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Proof.

Careful tilting of probability distributions. □

First-Order Asymptotics

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Corollary (First-Order Asymptotics on Rate)

If the sequence of distance measures satisfies

$$\sup_{n\in\mathbb{N}} \max_{x^n,\hat{x}^n} \frac{1}{n}\mu(x^n,\hat{x}^n) < \infty,$$

then we have

$$\limsup_{n\to\infty} R_n^*(\delta) = \limsup_{n\to\infty} I_{X^n}(\delta), \quad \textit{and}$$

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$$M^*(d) \le \inf_{\epsilon>0} \frac{|\mathcal{X}|^n}{\left|\mathcal{B}_{(d-\epsilon)/2}(\mathbf{0})\right|} \le \frac{|\mathcal{X}|^n}{\left|\mathcal{B}_{\lfloor (d-1)/2 \rfloor}(\mathbf{0})\right|}$$

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&\overset{\text{CS}}{\ge} \left(\sum_{\mathbf{x}} \sum_{\mathbf{y}\in\mathcal{B}_e(\mathbf{x})} P_{\mathbf{X}}(\mathbf{y}) \right)^2 = \frac{|\mathcal{B}_e(\mathbf{0})|^2}{|\mathcal{X}|^n}
\end{aligned}$$

Related Work

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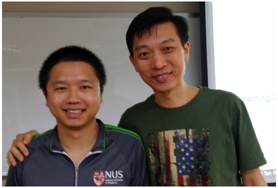

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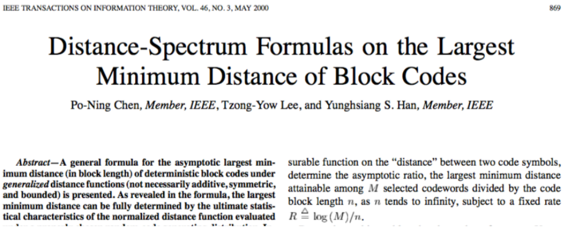
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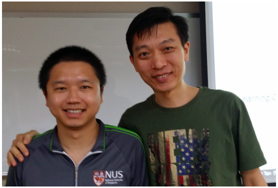

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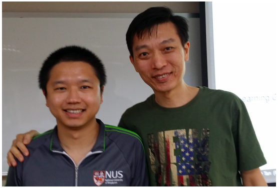

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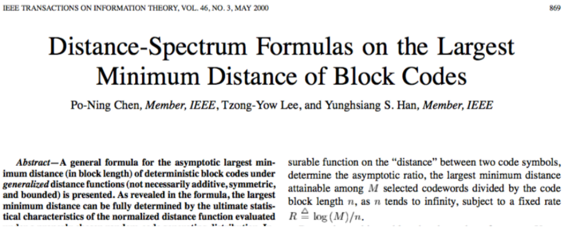
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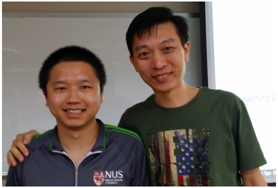

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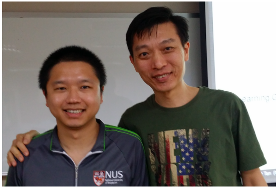

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- To appear in the IEEE Transactions on Information Theory in 2019.

Thanks!

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My collaborators and I at ITW 2017 (Kaohsiung)