The ε-Capacity Region of AWGN Multiple Access Channels with Feedback

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(Joint work with Lan V. Truong and Silas L. Fong)

National University of Singapore (NUS)

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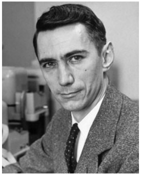
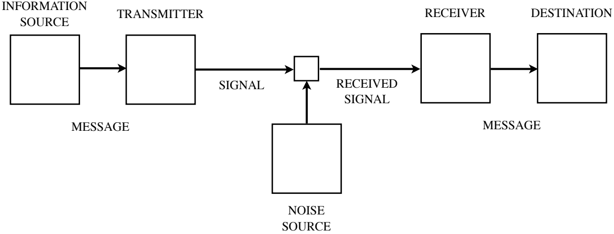
- Shannon Centenary:





Information Transmission

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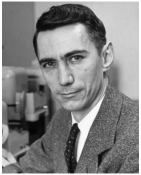

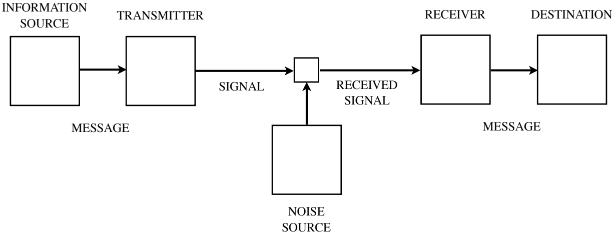

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$$C = \max_{P \in \mathcal{P}(\mathcal{X})} I(X;Y)$$

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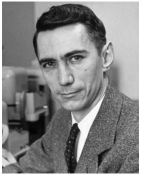

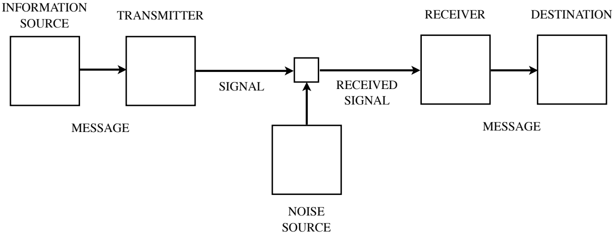

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- "Feedback doesn't increase capacity" *[Shannon (1956)]*

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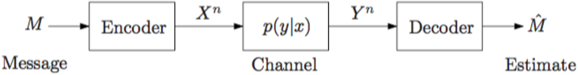

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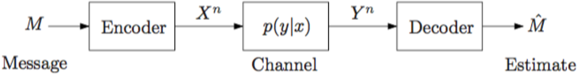

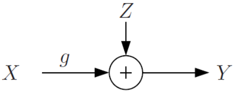

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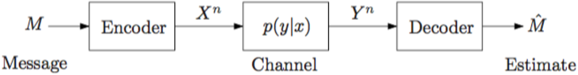

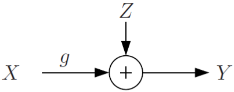

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$$Y_i = gX_i + Z_i, \qquad Z_i \sim \mathcal{N}(0, 1)$$

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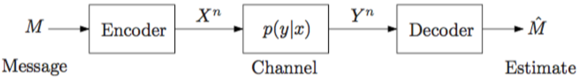

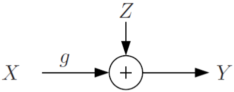

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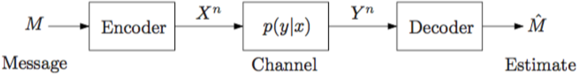

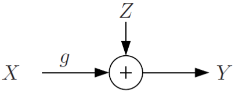

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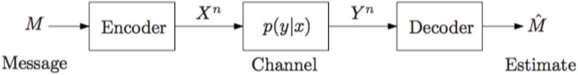

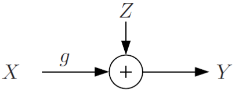

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- Expected or Long-Term power constraint

$$\frac{1}{\mathsf{M}}\sum_{m=1}^{\mathsf{M}} \left(\frac{1}{n}\sum_{i=1}^{n} X_i^2(m)\right) \le P.$$

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First-Order Results

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$$\lim_{\varepsilon\downarrow 0}\ \underline{\lim}_{n\to\infty}\ \frac{1}{n}\log \mathsf{M}^*_{\mathrm{PP}}(n,P,\varepsilon) = \mathsf{C}(P),$$
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- In n channel uses, can send up to $n\mathsf{C}(P)$ nats over $p(y|x)$ reliably.

First-Order Results

- If we do not demand that the avg error prob. vanishes *[Yoshihara (1964), Polyanskiy-Poor-Verdú (2010)]*,

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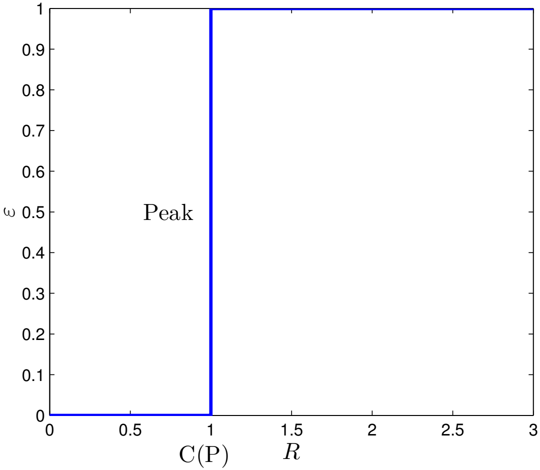
- Since for long-term, the ε-capacity depends on ε, the strong converse does not hold

Strong Converse?

$$\varepsilon = \varlimsup_{n\to\infty} \mathrm{P}_{\mathrm{e}}^{(n)}, \qquad R = \varliminf_{n\to\infty} \frac{1}{n} \log \mathsf{M}$$



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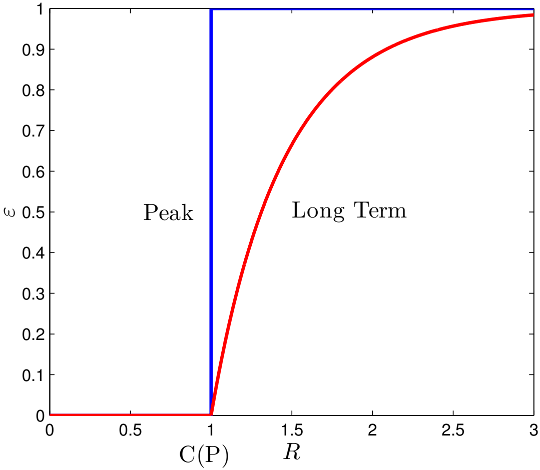

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$$\log \mathsf{M}^*_{\mathrm{PP}}(n,P,\varepsilon) = n\mathsf{C}(P) + \sqrt{n\mathsf{V}(P)}\Phi^{-1}(\varepsilon) + \frac{1}{2}\log n + O(1)$$

where the channel dispersion is

$$\mathsf{V}(x) := \frac{x(x+2)}{2(x+1)^2} \quad \text{squared nats per ch. use}$$

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$$\Phi(a) := \int_{-\infty}^{a} \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-t^2/2}\, \mathrm{d}t.$$

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- Second-order *[Yang-Caire-Durisi-Polyanskiy (2015)]*

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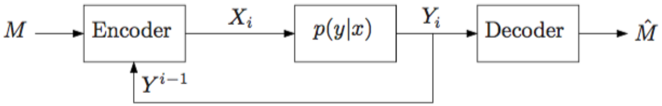

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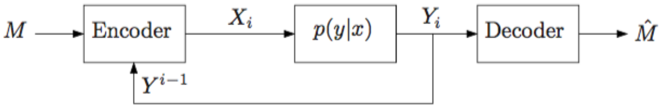

- Feedback helps to simplify coding schemes
- Long-term power constraint under feedback

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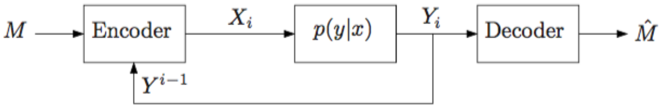

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Feedback : Existing Results

- First-order *[Shannon (1956)]*

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- Error exponent is infinity
- Suggests that the fixed-error results can also be drastically improved

AWGN Channels with Feedback : New Results

Theorem (Truong-Fong-T. (ISIT 2016))

For the direct part,

$$\log \mathsf{M}^*_{\mathrm{FB}}(n, P, \varepsilon) \geq n\mathsf{C}\left(\frac{P}{1-\varepsilon}\right) - \log\log n + O(1).$$

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From these results, the ε-capacity is

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AWGN Channels with Feedback : Remarks

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- With feedback, second-order term is at least

$$-\log\log n+O(1).$$

This is a great improvement over without feedback where the second-order term is *[Yang-Caire-Durisi-Polyanskiy (2015)]*

$$-\sqrt{\mathsf{V}\Big(\frac{P}{1-\varepsilon}\Big)}\sqrt{n\log n}+o(\sqrt{n}).$$

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- Hence,

$$\mathrm{P}_{\mathrm{e}}^{(n)} = \Pr(\mathcal{A}_1)\mathrm{P}_{\mathrm{e}}^{(n)}(\mathcal{A}_1) + \Pr(\mathcal{A}_2)\mathrm{P}_{\mathrm{e}}^{(n)}(\mathcal{A}_2) \le \varepsilon \cdot 1 + (1-\varepsilon)\frac{1}{n} \approx \varepsilon.$$

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- Convert expected long-term power to a peak-power code.
- Key observation

$$\exists \text{ LT-FB code } \{X_i(\cdot,\cdot)\}_{i=1}^n \text{ with M msges and } \mathrm{P}_\mathrm{e}^{(n)} \le \varepsilon$$

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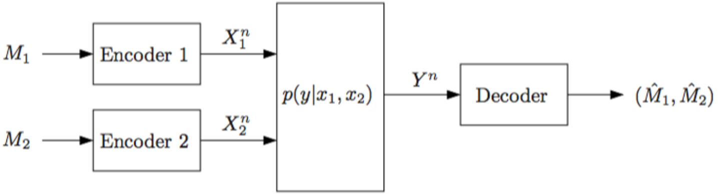
- Exploit connection between binary hypothesis testing and channel coding with feedback under peak-power constraint *[Polyanskiy-Poor-Verdú (2011)] [Fong-T. (2015)]*

MACs and Gaussian MACs

- The multiple access channel (MAC)



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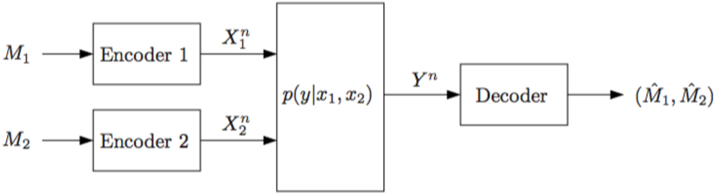
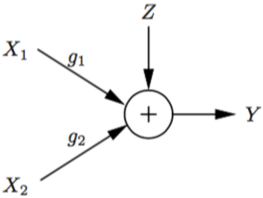
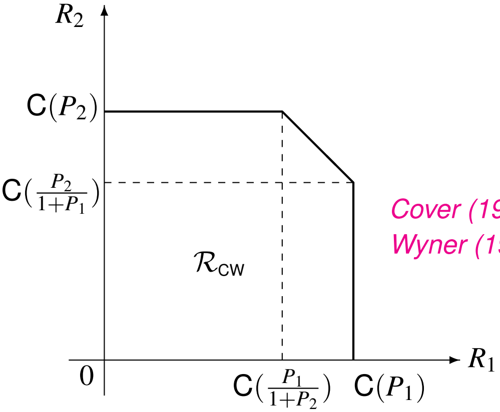


- The Gaussian multiple access channel



Again assume $g_1 = g_2 = 1$.

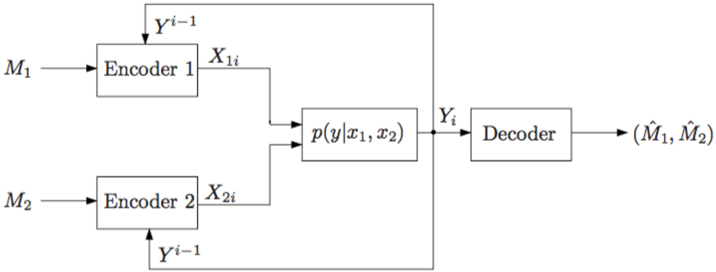
Capacity Region for the Gaussian MAC

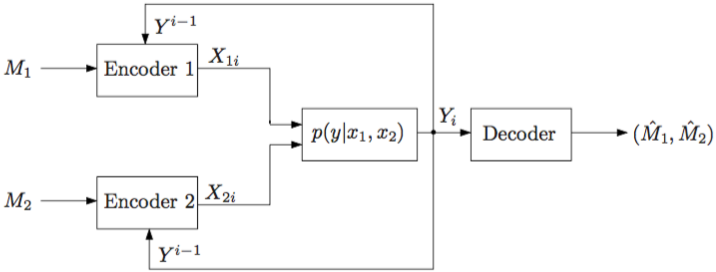


Cover (1975)
Wyner (1974)

$$R_1 \le \mathsf{C}(P_1)$$
$$R_2 \le \mathsf{C}(P_2)$$
$$R_1 + R_2 \le \mathsf{C}(P_1 + P_2)$$

Gaussian MAC with Feedback





Consider Gaussian version with expected long-term power constraints

$$\frac{1}{n}\sum_{i=1}^{n}\mathbb{E}\left[X_{1i}^2(M_1, Y^{i-1})\right] \le P_1, \quad \frac{1}{n}\sum_{i=1}^{n}\mathbb{E}\left[X_{2i}^2(M_2, Y^{i-1})\right] \le P_2.$$

Capacity Region of the G-MAC with Feedback

- *Ozarow (1984)* showed that the capacity region is

$$\mathcal{R}_{\mathrm{Ozarow}}(P_1, P_2) := \bigcup_{0\le\rho\le 1} \left\{ (R_1, R_2) \;\middle|\; \begin{array}{l} R_1 \le \mathsf{C}\big((1-\rho^2)P_1\big), \\ R_2 \le \mathsf{C}\big((1-\rho^2)P_2\big), \\ R_1 + R_2 \le \mathsf{C}\Big(P_1 + P_2 + 2\rho\sqrt{P_1 P_2}\Big) \end{array} \right\}.$$

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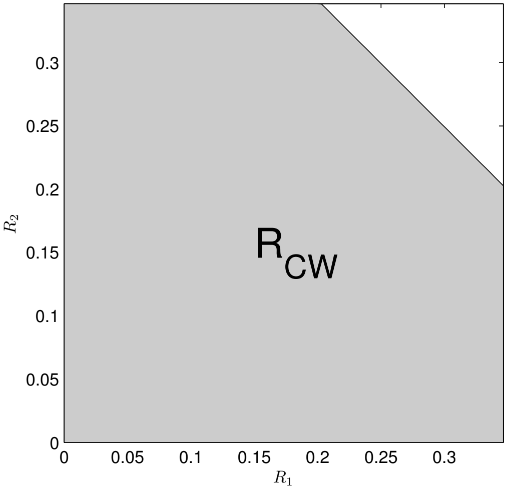
- Direct part is an extension of the Schalkwijk and Kailath coding scheme

CR of the G-MAC with Feedback $P_1 = P_2 = 1$



No feedback

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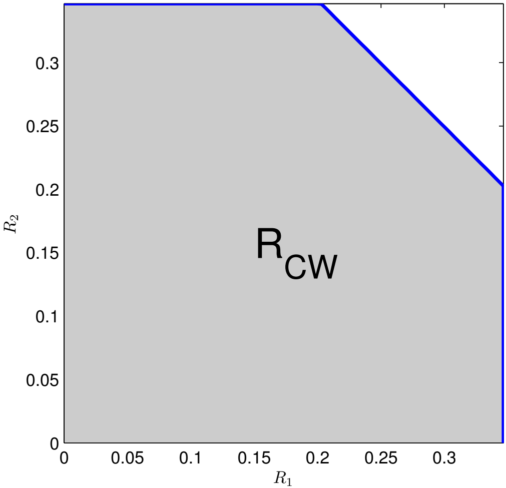

$\rho = 0$

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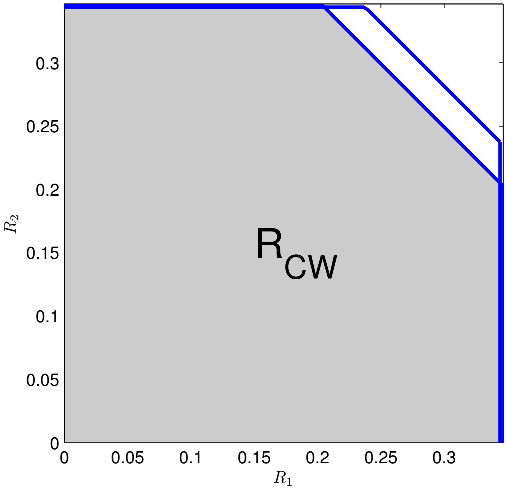

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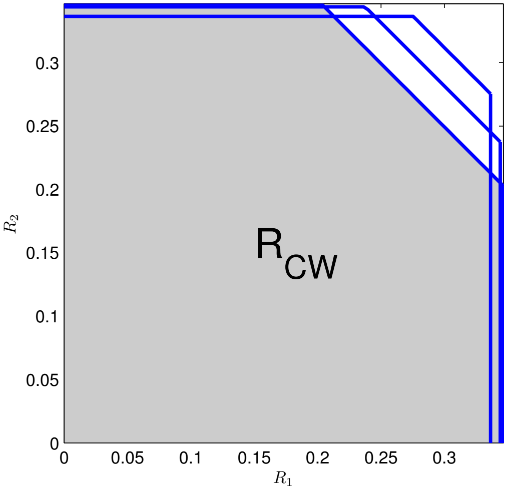
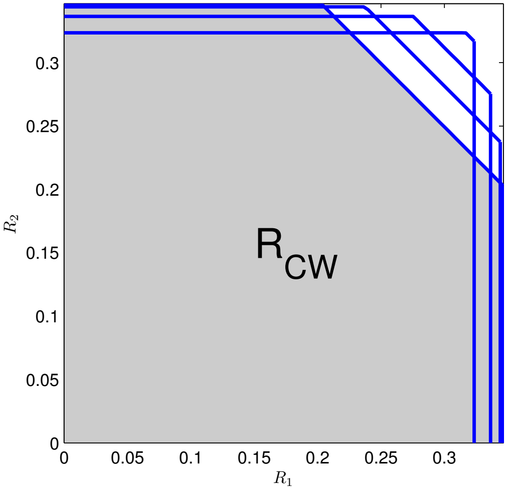


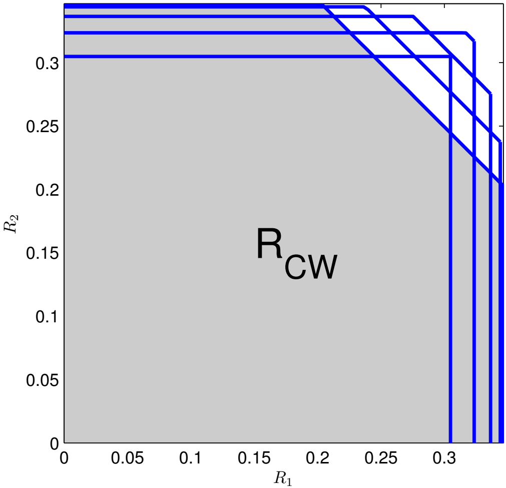
$\rho = 0.2$

CR of the G-MAC with Feedback $P_1 = P_2 = 1$



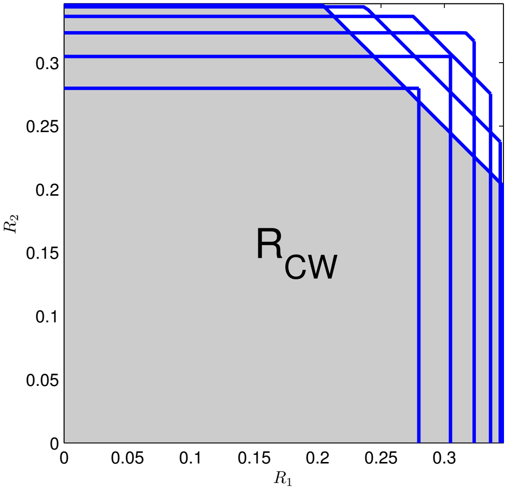
$$\rho = 0.3$$

CR of the G-MAC with Feedback $P_1 = P_2 = 1$



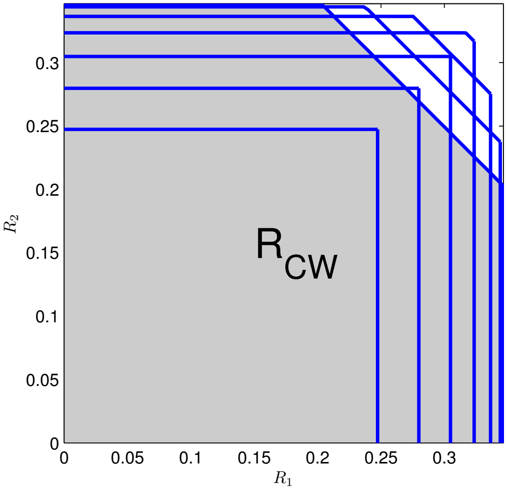
$$\rho = 0.4$$

CR of the G-MAC with Feedback $P_1 = P_2 = 1$



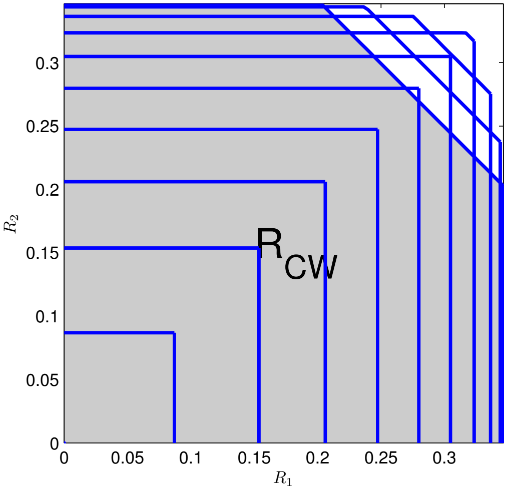
$\rho = 0.5$

CR of the G-MAC with Feedback $P_1 = P_2 = 1$



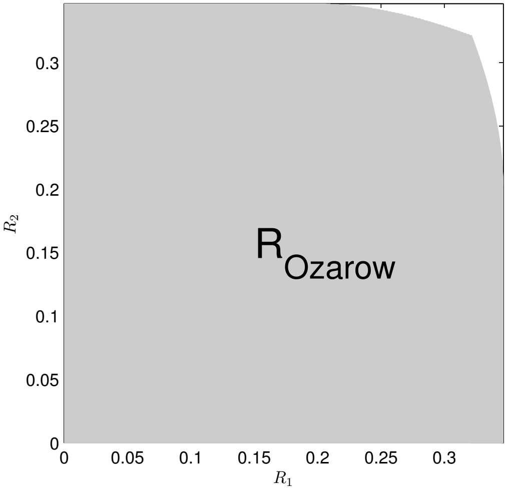
$\rho = 0.6$

CR of the G-MAC with Feedback $P_1 = P_2 = 1$



$\rho = 1.0$

CR of the G-MAC with Feedback $P_1 = P_2 = 1$



The Ozarow region

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  $\iff \exists$ sequence of codes with $(\mathsf{M}_1, \mathsf{M}_2)$ messages s.t.

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  and the average probability of error

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- $\mathcal{C}_\varepsilon(P_1, P_2)$ is the set of all ε-achievable (R_1, R_2).

ε-Capacity Region of the G-MAC with Feedback

Theorem (Truong-Fong-T. (arXiv 2015))

The ε-capacity region is

$$\mathcal{C}_\varepsilon(P_1, P_2) = \mathcal{R}_{\text{Ozarow}}\left(\frac{P_1}{1-\varepsilon}, \frac{P_2}{1-\varepsilon}\right), \quad \textit{for all} \quad \varepsilon \in [0, 1).$$

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If we can tolerate an error of $\le \varepsilon$, we can operate at (R_1,R_2) satisfying

$$R_1\le \mathsf{C}\left(\frac{(1-\rho^2)P_1}{1-\varepsilon}\right)$$

$$R_2\le \mathsf{C}\left(\frac{(1-\rho^2)P_2}{1-\varepsilon}\right),\qquad \text{for any } 0\le\rho\le 1.$$

$$R_1+R_2\le \mathsf{C}\left(\frac{P_1+P_2+2\rho\sqrt{P_1P_2}}{1-\varepsilon}\right)$$

This is optimal.

ε-Capacity of the G-MAC with Feedback : Remarks

- $\varepsilon = 0$ recovers Ozarow's result

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$$\log \mathsf{M}_1 \le \gamma_1 - \log^+ \left[1 - \varepsilon - \Pr\left(\sum_{i=1}^n \log \frac{W(Y_i|X_{1i}, X_{2i})}{Q_{Y_i|X_{2i}}(Y_i|X_{2i})} \ge \gamma_1 \right) \right]$$

$$\log \mathsf{M}_2 \le \gamma_2 - \log^+ \left[1 - \varepsilon - \Pr\left(\sum_{i=1}^n \log \frac{W(Y_i|X_{1i}, X_{2i})}{Q_{Y_i|X_{1i}}(Y_i|X_{1i})} \ge \gamma_2 \right) \right]$$

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Lemma (“Single-Letterization”)

$$|\rho| \le 1,$$

$$\sum_{i=1}^n \left(P_{1i}(1-\rho_i^2)\right) \le nP_1(1-\rho^2), \quad \textit{and}$$

$$\sum_{i=1}^n \left(P_{1i} + P_{2i} + 2\rho_i\sqrt{P_{1i}P_{2i}}\right) \le n\left(P_1 + P_2 + 2\rho\sqrt{P_1P_2}\right).$$

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Then, with a good choice of Q's

$$\Pr\left(\sum_{i=1}^{n} \log \frac{W(Y_i|X_{1i}, X_{2i})}{Q_{Y_i|X_{2i}}(Y_i|X_{2i})} \geq \gamma_1\right) \leq \frac{1}{T} + O(n^{-1/3})$$
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$$\frac{1}{T} = 1 - \varepsilon - O(n^{-1/3}) \quad \text{so} \quad \gamma_1 = n\mathsf{C}\left(\frac{P_1(1-\rho^2)}{1-\varepsilon} \right) + O(n^{2/3}).$$

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- By product: Second-order term is upper bounded by $O(n^{2/3})$.

Wrap Up

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- Generalized a result by *Ozarow (1984)* to non-vanishing $\varepsilon \in [0,1)$
- Established ε-capacity region for AWGN-MAC with feedback

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- `http://arxiv.org/abs/1512.05088`

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