Asymptotic Estimates in Information Theory with Non-Vanishing Error Probabilities

Vincent Y. F. Tan

Dept. of ECE and Dept. of Mathematics
National University of Singapore (NUS)

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Acknowledgements

This is joint work with

1. Sy-Quoc Le and Mehul Motani (NUS)

2. Jon Scarlett (Cambridge, now at EPFL)

Outline

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1. Motivation, Background and History
2. Gaussian Interference Channel with Very Strong Interference
3. Gaussian MAC with Degraded Message Sets

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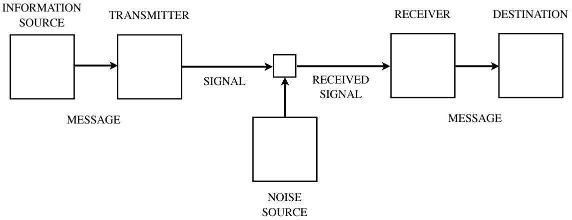
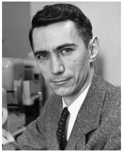
4. Conclusion

Transmission of Information





Shannon's Figure 1

- Information theory $\equiv$ Finding fundamental limits for reliable information transmission

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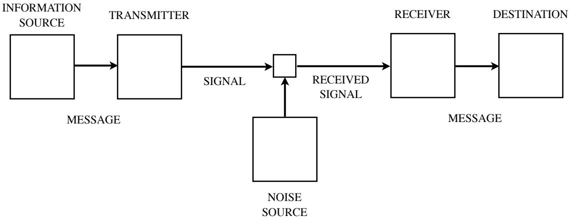
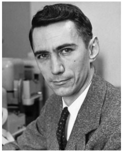
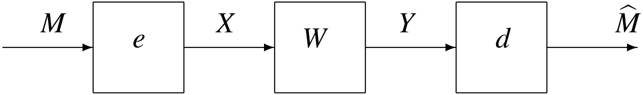




Shannon's Figure 1

- Information theory $\equiv$ Finding fundamental limits for reliable information transmission
- Channel coding: Concerned with the maximum rate of communication in bits/channel use

Channel Coding (One-Shot)



- A code is an triple $\mathcal{C} = \{\mathcal{M}, e, d\}$ where $\mathcal{M}$ is the message set and $b(e(m)) \leq S$ for some cost function $b(\cdot)$ and cost S

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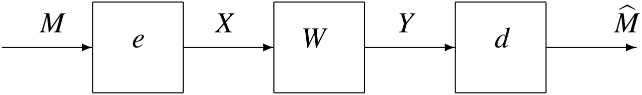

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$$p_{\mathrm{err}}(\mathcal{C}) \ := \ \Pr\left[\widehat{M} \neq M\right]$$

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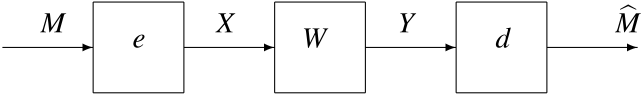

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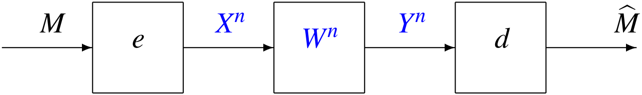
- A non-asymptotic fundamental limit can be defined as

$$M^*(W, \varepsilon, S) := \sup\left\{m \in \mathbb{N} \,\middle|\, \exists\, \mathcal{C} \text{ s.t. } m = |\mathcal{M}|,\, p_{\mathrm{err}}(\mathcal{C}) \le \varepsilon\right\}$$

Channel Coding (n-Shot) for AWGN



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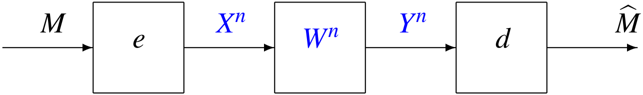

- Consider n independent uses of an additive white Gaussian channel W^n
- For vectors $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and $\mathbf{y} := (y_1, \ldots, y_n) \in \mathbb{R}^n$, with

$$\|\mathbf{x}\|_2^2 \le nS,$$

the channel law is

$$W^n(\mathbf{y}|\mathbf{x}) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(y_i - x_i)^2}{2}\right)$$

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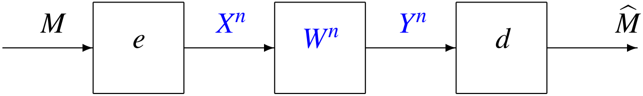

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- Non-asymptotic fundamental limit for n uses of W

$$M^*(W^n, \varepsilon, S)$$

Single-User Asymptotic Evaluation

Theorem (Hayashi (2009), Polyanskiy-Poor-Verdú (2010), Tan-Tomamichel (2014))

For every $\varepsilon \in (0,1)$*, we have*

$$\log M^*(W^n, \varepsilon, S) = n\mathrm{C}(S) + \sqrt{n\mathrm{V}(S)}\Phi^{-1}(\varepsilon) + \frac{1}{2}\log n + O(1)$$

where $\mathrm{C}(S)$ *and* $\mathrm{V}(S)$ *are the capacity and dispersion defined as*

$$\mathrm{C}(S) = \frac{1}{2}\log(1+S), \qquad \mathrm{V}(S) = \frac{S(S+2)}{(S+1)^2}\log^2 e$$

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M. Hayashi



Polyanskiy-Poor-Verdú

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Single-User Asymptotic Evaluation: Interpretation

$$R^*(W^n, \varepsilon, S) := \frac{\log M^*(W^n, \varepsilon, S)}{n} = \underbrace{\mathrm{C}(S) + \sqrt{\frac{\mathrm{V}(S)}{n}}\Phi^{-1}(\varepsilon)}_{\text{Gaussian approximation}} + O\left(\frac{\log n}{n}\right)$$

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- Interpretation: The backoff from $\mathrm{C}(S)$ at finite blocklength n and tolerable error probability is approximately

$$\sqrt{\frac{\mathrm{V}(S)}{n}}\,\Phi^{-1}(1-\varepsilon)$$

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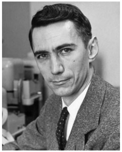
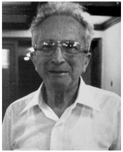
- Small ε implies large backoff
- Can compare to actual finite blocklength bounds
- Gaussian approximation is good for some n and ε

Background: Shannon's Channel Coding Theorem

- Shannon's noisy channel coding theorem and
- Shannon's (1959), Yoshihara's (1964) and Wolfowitz's (1978) strong converse state that





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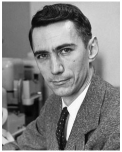

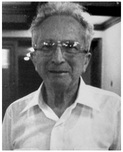

Theorem (Shannon (1949), Shannon (1959))

$$\lim_{n \to \infty} \frac{1}{n} \log M^*(W^n, \varepsilon, S) = \mathrm{C}(S), \qquad \forall\, \varepsilon \in (0, 1)$$

Background: Shannon's Channel Coding Theorem

$$\lim_{n\to\infty} \frac{1}{n} \log M^*(W^n, \varepsilon, S) = \mathrm{C}(S) \quad \text{bits/channel use}$$

- Channel coding theorem for AWGN channels is independent of $\varepsilon \in (0,1)$

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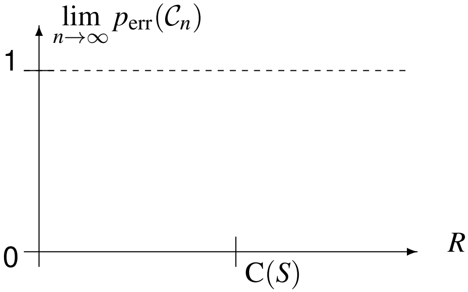

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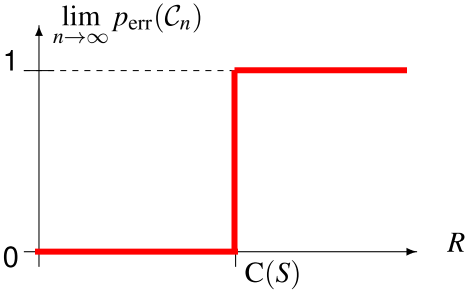

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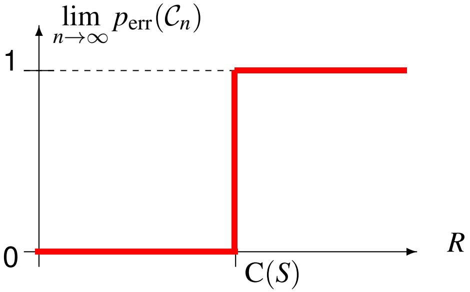

- Phase transition at capacity

Background: Second-Order Coding Rates

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- More precisely, what happens when

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  for some $L \in \mathbb{R}$?
- Here L is known as the second-order coding rate of the code
- Note that L can be negative (cf. Hayashi (2008), Hayashi (2009))

Background: Second-Order Coding Rates

Assume rate of the code satisfies

$$\frac{1}{n}\log|\mathcal{M}_n| = \mathrm{C}(S) + \frac{L}{\sqrt{n}} + o\Big(\frac{1}{\sqrt{n}}\Big)$$

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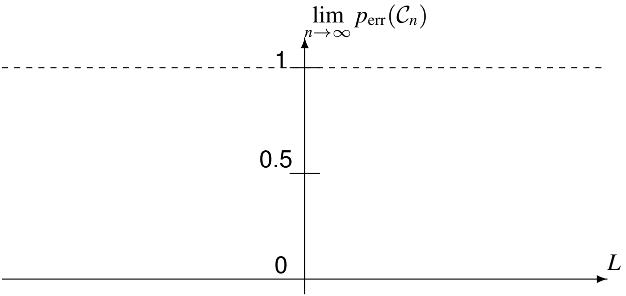

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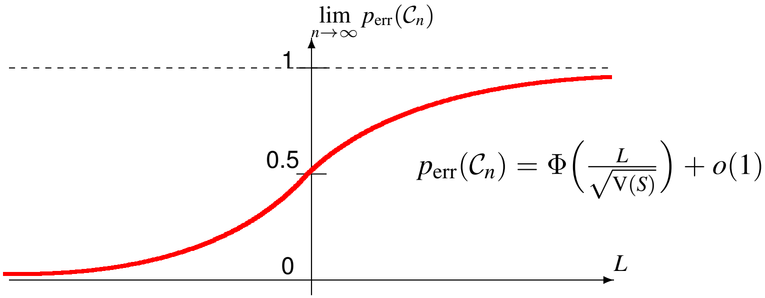

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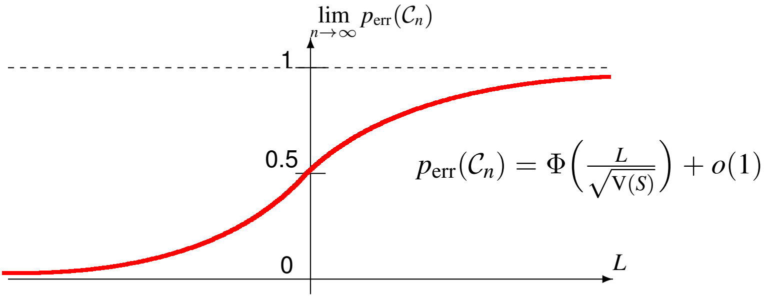
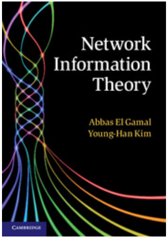


For an error probability ε, the optimum second-order coding rate is

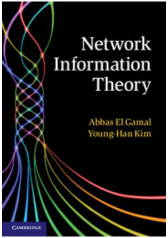
$$L^*(\varepsilon) := \sqrt{\mathrm{V}(S)}\,\Phi^{-1}(\varepsilon)$$



Network Information Theory
by A. El Gamal and Y.-H. Kim

- Many problems unsolved

Network Information Theory



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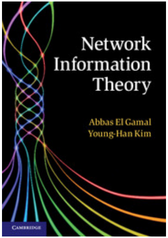

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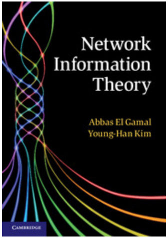

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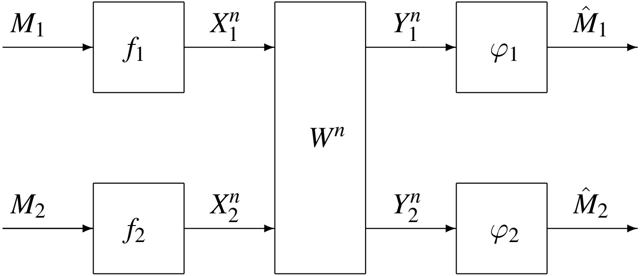
- Many problems unsolved
- My agenda is to understand second-order behavior of solved NIT problems
- Gain new insights on second-order optimal coding schemes
- We study two simple NIT problems in the rest of the talk

Outline

Gaussian Interference Channel



- Two-sender two-receiver GIC is given as

$$Y_{1i} = g_{11} X_{1i} + g_{12} X_{2i} + Z_{1i}, \qquad Y_{2i} = g_{21} X_{1i} + g_{22} X_{2i} + Z_{2i}$$

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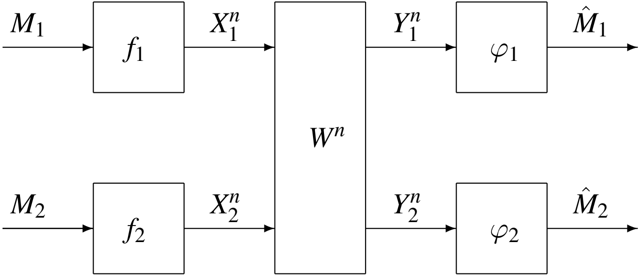

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- Channel inputs are power limited

$$\sum_{i=1}^{n} X_{ji}^2 \le nS_j, \qquad j = 1, 2$$

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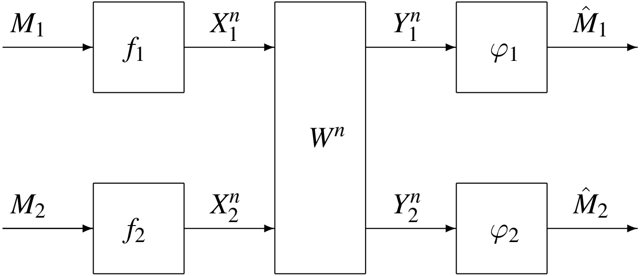

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- Capacity region is an open problem in NIT

Very Strong Inference

- Define the signal-to-noise ratios as

$$\mathrm{snr}_1 = g_{11}^2 S_1, \qquad \mathrm{snr}_2 = g_{22}^2 S_2$$

- Define the interference-to-noise ratios as

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- A GIC is said to have strictly very strong interference (SVSI) if

$$\mathrm{snr}_1 < \frac{\mathrm{inr}_2}{1+\mathrm{snr}_2}, \qquad \mathrm{snr}_2 < \frac{\mathrm{inr}_1}{1+\mathrm{snr}_1}$$

- Equivalently

$$\mathrm{C}(\mathrm{snr}_1) + \mathrm{C}(\mathrm{snr}_2) < \min\{\mathrm{C}(\mathrm{snr}_1 + \mathrm{inr}_1), \mathrm{C}(\mathrm{snr}_2 + \mathrm{inr}_2)\}$$

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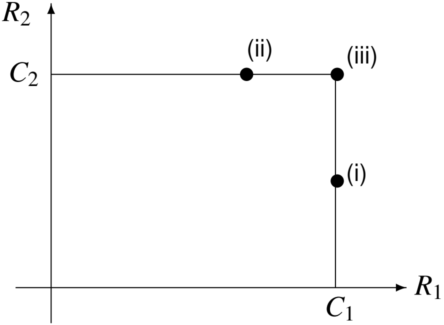
- Intuition: Receiver 1 decodes interference M_2, then uses that to decode intended message M_1 (and vice versa)

Capacity Region for GICs with SVSI



A. Carleial

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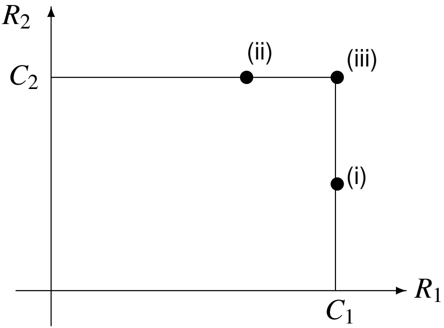

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$$R_1 \leq C_1 := \mathrm{C}(\mathrm{snr}_1) \qquad R_2 \leq C_2 := \mathrm{C}(\mathrm{snr}_2)$$

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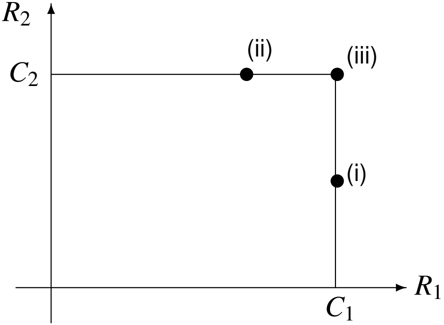

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- Examine deviations of order $\frac{1}{\sqrt{n}}$ away from the boundary
- Three distinct regions.

Second-Order Coding Rate Region

- (L_1, L_2) is an $(\varepsilon, R_1^*, R_2^*)$-achievable second-order coding rate pair if there exists a sequence of $(n, M_{1n}, M_{2n}, \varepsilon_n)$-codes such that the code sizes M_{jn} satisfy

$$\liminf_{n\to\infty} \frac{1}{\sqrt{n}}(\log M_{jn} - nR_j^*) \geq L_j, \qquad j = 1, 2$$

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- If (L_1, L_2) is $(\varepsilon, R_1^*, R_2^*)$-achievable, then there exists a sequence of codes such that

$$\log M_{jn} \geq nR_j^* + \sqrt{n}L_j + o(\sqrt{n})$$

and

$$\varepsilon_n \leq \varepsilon + o(1).$$

Optimum Second-Order Coding Rate Region

- The set of all $(\varepsilon, R_1^*, R_2^*)$-achievable second-order coding rate pairs is called the $(\varepsilon, R_1^*, R_2^*)$-optimum second-order coding rate region

$$\mathcal{L}(\varepsilon; R_1^*, R_2^*)$$

Optimum Second-Order Coding Rate Region

- The set of all $(\varepsilon, R_1^*, R_2^*)$-achievable second-order coding rate pairs is called the $(\varepsilon, R_1^*, R_2^*)$-optimum second-order coding rate region

$$\mathcal{L}(\varepsilon; R_1^*, R_2^*)$$

- Note: In the single-user case,

$$L^*(\varepsilon) := \sup \mathcal{L}(\varepsilon; R_1^* = C_1) = \sqrt{\mathrm{V}(S_1)}\Phi^{-1}(\varepsilon)$$

- This follows from Hayashi's work (2009)

Characterize $\mathcal{L}(\varepsilon; R_1^*, R_2^*)$

- This is joint work with

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M. Motani (NUS)

- We want to characterize $\mathcal{L}(\varepsilon; R_1^*, R_2^*)$ for all points (R_1^*, R_2^*) on the boundary of the capacity region

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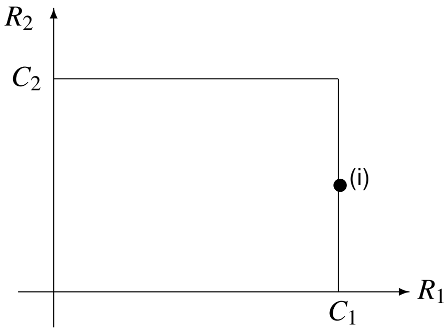

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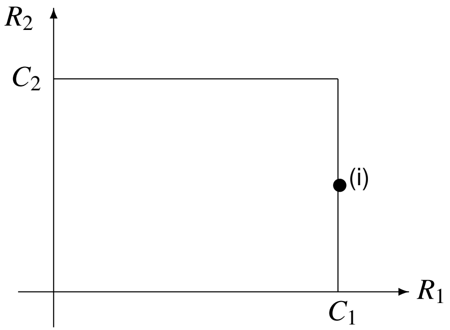

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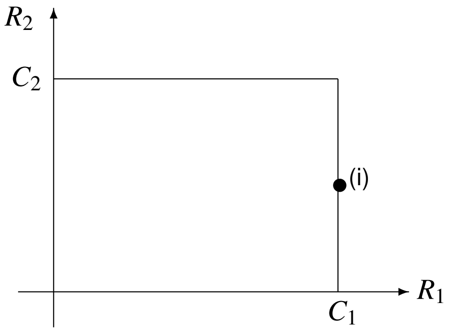
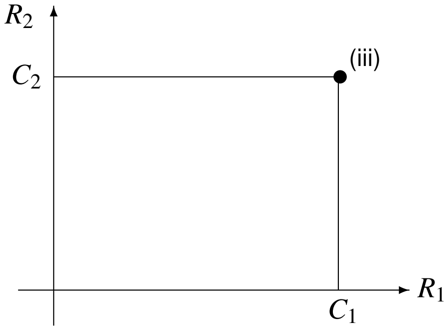


- In case (i), we are far from the horizontal boundary $R_2 < C_2$
- Error event for user 2 vanishes (large deviations)
- Hence second-order asymptotics only pertains to user 1

$$\mathcal{L}(\varepsilon; R_1^*, R_2^*) = \left\{ (L_1, L_2) : L_1 \le \sqrt{V_1}\Phi^{-1}(\varepsilon) \right\}.$$

Main Result: Corner Point



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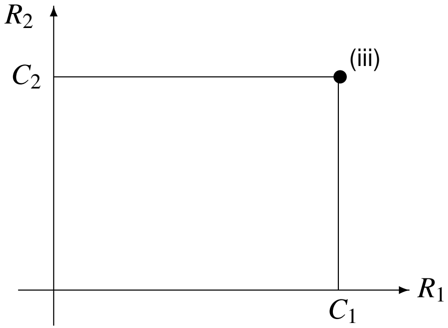

- In case (iii), we are operating near both boundaries
- So both constraints are active and we see L_1 and L_2 in the optimum second-order coding rate region is

$$\mathcal{L}(\varepsilon; R_1^*, R_2^*) = \left\{ (L_1, L_2) : \Phi\left(-\frac{L_1}{\sqrt{V_1}} \right) \Phi\left(-\frac{L_2}{\sqrt{V_2}} \right) \geq 1 - \varepsilon \right\}.$$

Heuristic Derivation of Case (iii)

- Let $\mathcal{G}_j := \{\hat{M}_j = \hat{M}\}$ be the event that message $j = 1, 2$ is decoded correctly

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- But we know from the single-user result that the error probability

$$\Pr(\mathcal{G}_j^c) \approx \Phi\Big(\frac{L_j}{\sqrt{V_j}}\Big) \quad \Longrightarrow \quad \Pr(\mathcal{G}_j) \approx \Phi\Big(-\frac{L_j}{\sqrt{V_j}}\Big).$$

  So the set of all (ε, C_1, C_2)-achievable second-order coding rates is

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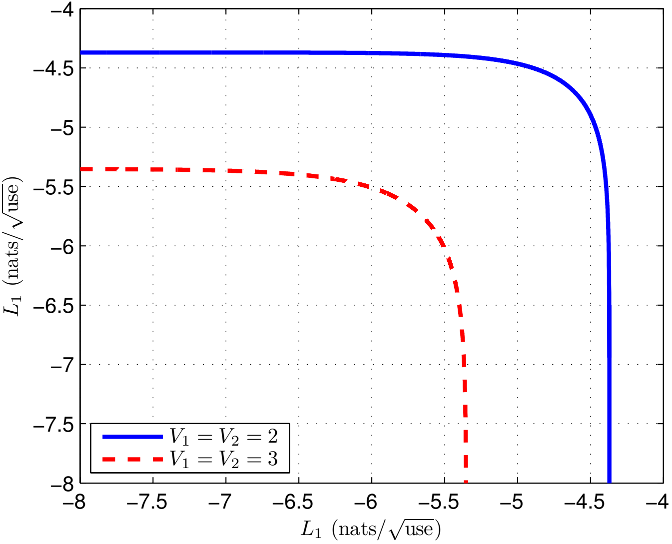

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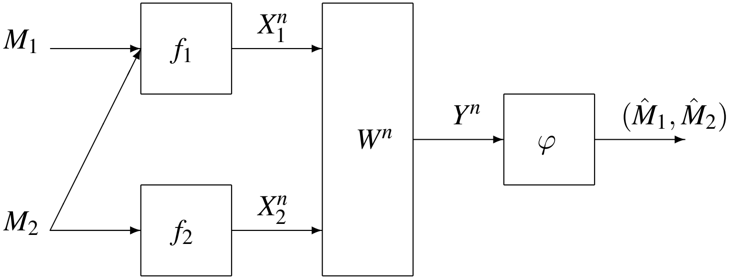
- Intuition that the GIC with SVSI is analogous to 2 independent direct channels carries over, even in the finer second-order sense

Outline

Gaussian MAC with Degraded Message Sets



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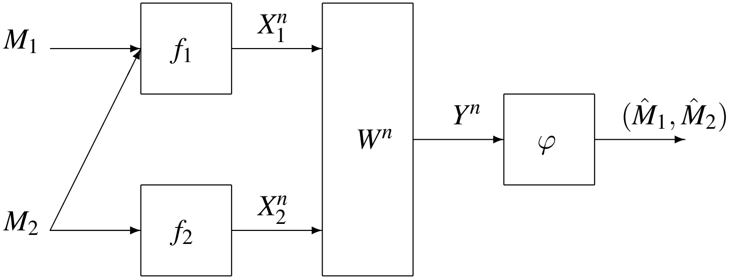

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- Channel inputs are power limited

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- Uses superposition coding (Cover (1972))

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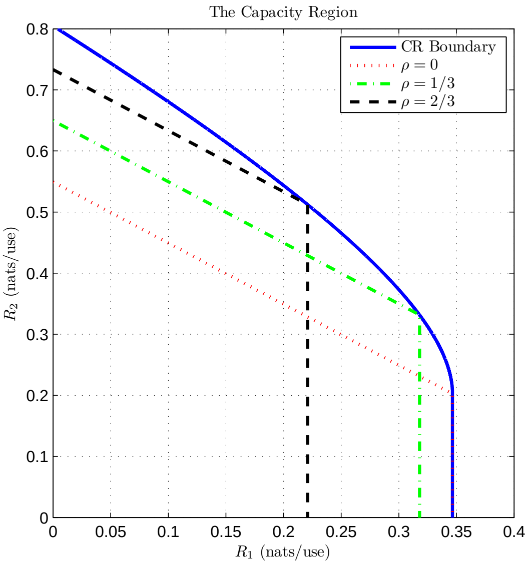

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Some Basic Definitions

- Mutual informations

$$\mathbf{I}(\rho) := \begin{bmatrix} I_1(\rho) \\ I_{12}(\rho) \end{bmatrix} = \begin{bmatrix} \mathrm{C}\big((1-\rho^2)S_1\big) \\ \mathrm{C}\big(S_1 + S_2 + 2\rho\sqrt{S_1 S_2}\big) \end{bmatrix}$$

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- Dispersions $\mathrm{V}(x,y) := \frac{x(y+2)}{2(x+1)(y+1)}$ and $\mathrm{V}(x) := \mathrm{V}(x,x)$

$$\mathbf{V}(\rho) := \begin{bmatrix} V_1(\rho) & V_{1,12}(\rho) \\ V_{1,12}(\rho) & V_{12,12}(\rho) \end{bmatrix}$$

where

$$V_1(\rho) := \mathrm{V}\big((1-\rho^2)S_1\big), \quad V_{12,12}(\rho) := \mathrm{V}\big(S_1 + S_2 + 2\rho\sqrt{S_1S_2}\big)$$

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Generalization of Inverse CDF of a Gaussian

- For a positive semi-definite matrix $\mathbf{V}$,

$$\Psi(z_1, z_2, \mathbf{V}) = \int_{-\infty}^{z_1} \int_{-\infty}^{z_2} \mathcal{N}(\mathbf{0}, \mathbf{V}) \, \mathrm{d}\mathbf{u}$$

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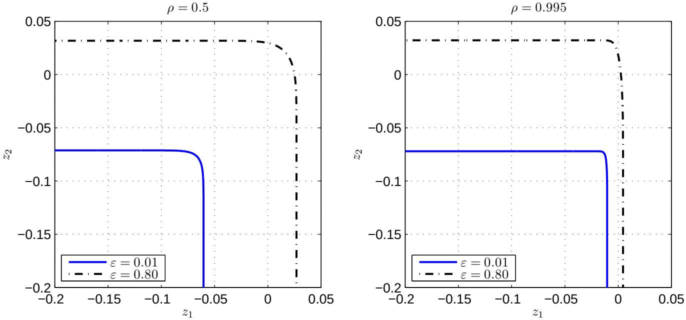
- Given $\varepsilon \in (0, 1)$,

$$\Psi^{-1}(\mathbf{V}, \varepsilon) = \{(z_1, z_2) : \Psi(-z_1, -z_2, \mathbf{V}) \geq 1 - \varepsilon\} .$$



The Main Result: Vertical Boundary

Points on vertical boundary reduce to scalar dispersion as sum rate constraint is in error exponents regime

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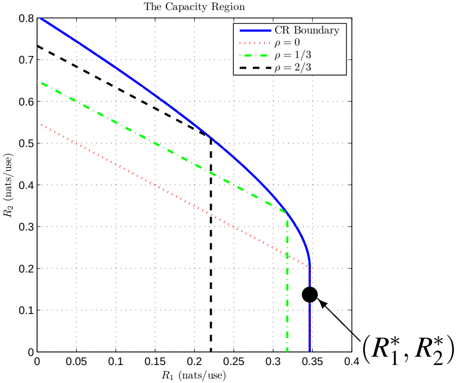

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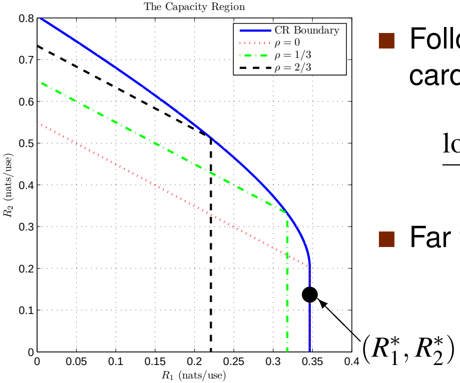

- Following expansion holds for cardinality of first codebook

$$\frac{\log M_{1n}}{n} \approx I_1(0) + \sqrt{\frac{V_1(0)}{n}}\Phi^{-1}(\varepsilon)$$

- Far from sum rate constraint

$$\lim_{n\to\infty} \frac{1}{n} \log(M_{1n}M_{2n}) < I_{12}(0)$$

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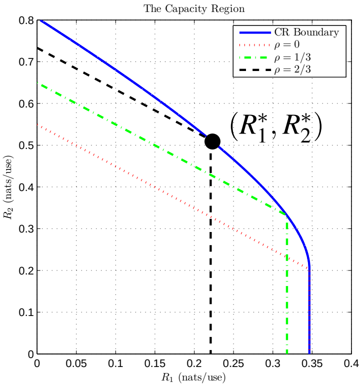
Different behavior in the curved region

$$\mathcal{L}(\varepsilon; R_1^*, R_2^*) = \left\{ (L_1, L_2) : \begin{bmatrix} L_1 \\ L_1 + L_2 \end{bmatrix} \in \bigcup_{\beta \in \mathbb{R}} \beta \mathbf{D}(\rho) + \Psi^{-1}(\mathbf{V}(\rho), \varepsilon) \right\}$$



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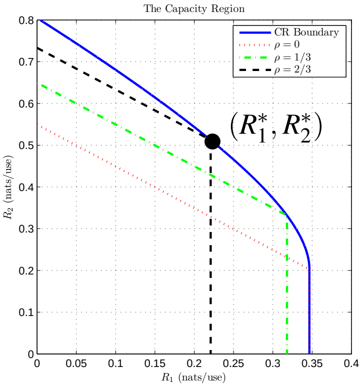

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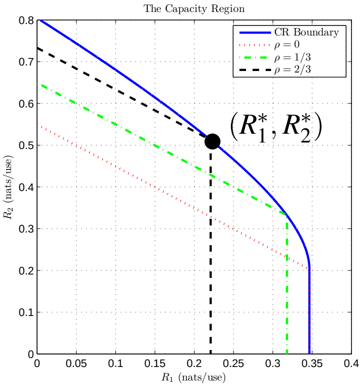

- $\mathbf{D}(\rho)$ doesn't appear usually
- $\Psi^{-1}(\mathbf{V}(\rho), \varepsilon)$: corresponds using Gaussian with covariance matrix

$$\mathbf{\Sigma}(\rho) = \begin{bmatrix} S_1 & \rho\sqrt{S_1 S_2} \\ \rho\sqrt{S_1 S_2} & S_2 \end{bmatrix}$$

Achieving all Second-Order Rate Pairs

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- Non-empty regions in CR not in trapezium achievable by

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- Use the above distribution dependent on blocklength:

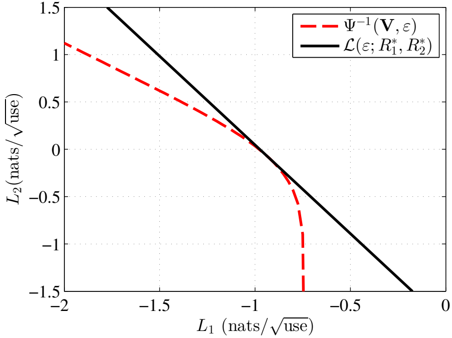
$$\rho_n = \rho + \frac{\beta}{\sqrt{n}}$$

- By a Taylor expansion,

$$\mathbf{I}(\rho_n) \approx \mathbf{I}(\rho) + (\rho_n - \rho)\mathbf{D}(\rho) = \mathbf{I}(\rho) + \frac{\beta \mathbf{D}(\rho)}{\sqrt{n}}$$

explaining the slope term.

Illustration of Second-Order Coding Rates



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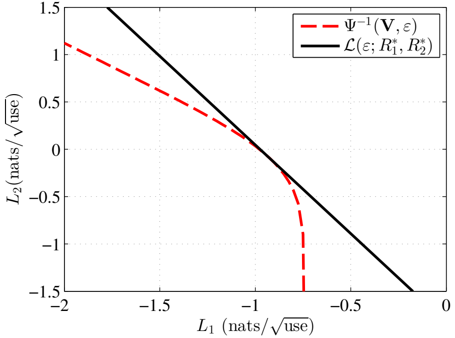

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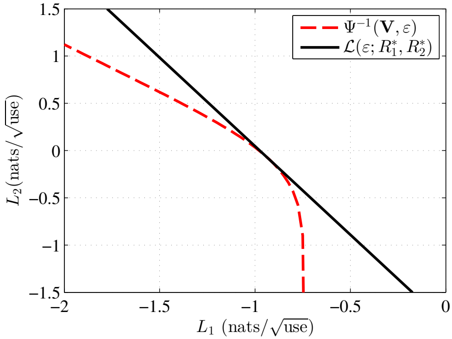

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- Second-order rates achieved using a single input distribution $\mathcal{N}(\mathbf{0}, \mathbf{\Sigma}(\rho))$ is not optimal
- Need to vary input distribution with blocklength to achieve all points in $\mathcal{L}(\varepsilon; R_1^*, R_2^*)$

Outline

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