Two Applications of the Gaussian Poincaré Inequality in the Shannon Theory

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Gaussian Poincaré Inequality

Theorem

For $Z^n \stackrel{iid}{\sim} \mathcal{N}(0,1)$ *and any differentiable mapping* f *such that*

$$\mathbb{E}[(f(Z^n))^2] < \infty, \quad \textit{and} \quad \mathbb{E}[\|\nabla f(Z^n)\|^2] < \infty$$

we have

$$\mathrm{var}[f(Z^n)] \le \mathbb{E}[\|\nabla f(Z^n)\|^2].$$

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- Controlling the variance of $f(Z^n)$ that is a function of i.i.d. random variables in terms of the gradient of $f(Z^n)$
- Using the Gaussian Poincaré inequality for appropriate f,

$$\mathrm{var}[f(Z^n)] = O(n).$$

Gaussian Poincaré Inequality in Shannon Theory

- Polyanskiy and Verdú (2014) bounded the KL divergence between the empirical output distribution of AWGN channel codes P_{Y^n} and the n-fold product of the CAOD P_Y^*, i.e.,

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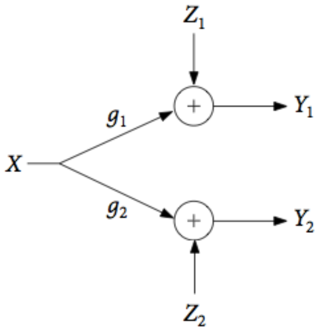
- Often we need to bound the variance of certain log-likelihood ratios (dispersion)
- Demonstrate its utility by establishing
  - Strong converse for the Gaussian broadcast channels
  - Properties of the empirical output distribution of delay-limited codes for quasi-static fading channels

Gaussian Broadcast Channel



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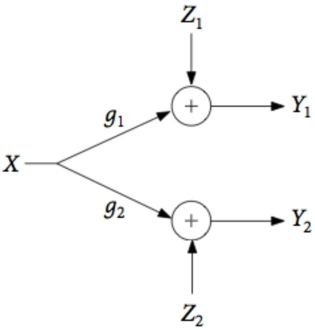

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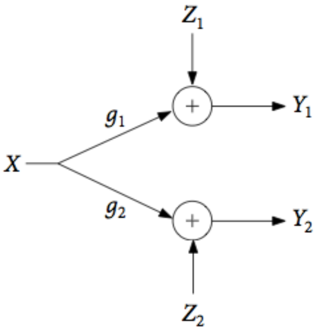

- Assume $g_1 = g_2 = 1$ and $\sigma_2^2 > \sigma_1^2$
- Input X^n must satisfy

$$\|X^n\|_2^2 = \sum_{i=1}^{n} X_i^2 \leq nP$$

Capacity Region

- An $(n, M_{1n}, M_{2n}, \varepsilon_n)$-code consists of
  - an encoder $f : \{1, \ldots, M_{1n}\} \times \{1, \ldots, M_{2n}\} \to \mathbb{R}^n$ such that the power constraint is satisfied;
  - two decoders $\varphi_j : \mathbb{R}^n \to \{1, \ldots, M_{jn}\}$ for $j = 1, 2$;

  such that the average error probability

$$P_e^{(n)} := \Pr(\hat{W}_1 \neq W_1 \text{ or } \hat{W}_2 \neq W_2) \leq \varepsilon_n.$$

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- (R_1, R_2) is achievable $\Leftrightarrow \exists$ a sequence of $(n, M_{1n}, M_{2n}, \varepsilon_n)$-codes s.t.

$$\liminf_{n\to\infty} \frac{1}{n} \log M_{jn} \geq R_j, \quad j = 1, 2, \quad \text{and}$$

$$\lim_{n\to\infty} \varepsilon_n = 0.$$

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$$\mathcal{C} = \mathcal{R}_{\mathrm{BC}} = \bigcup_{\alpha \in [0,1]} \mathcal{R}(\alpha)$$

where

$$\mathcal{R}(\alpha) = \left\{ (R_1, R_2) : R_1 \le \mathrm{C}\left(\frac{\alpha P}{\sigma_1^2}\right), R_2 \le \mathrm{C}\left(\frac{(1-\alpha)P}{\alpha P + \sigma_2^2}\right) \right\}$$

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- Converse part: Fano's inequality + Entropy power inequality

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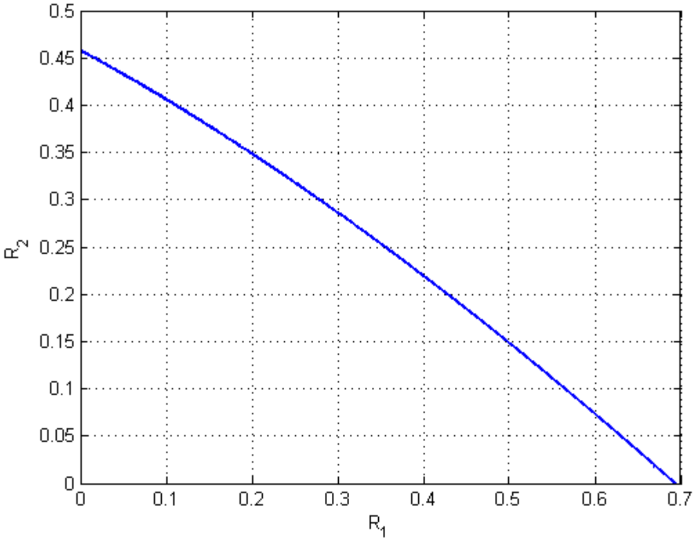

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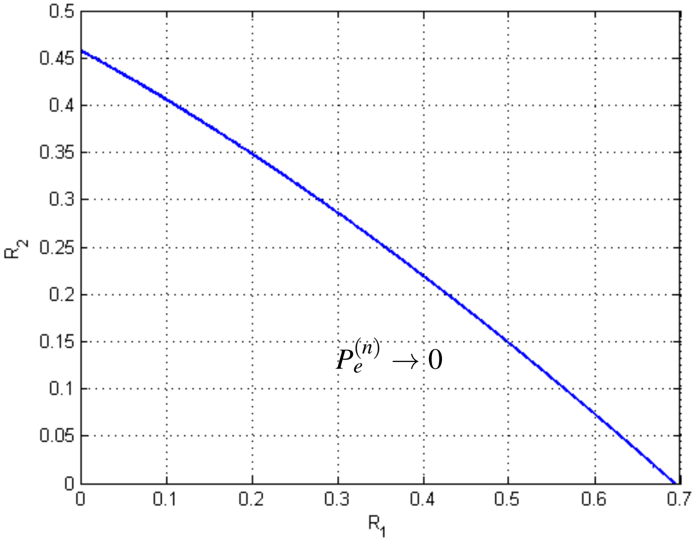

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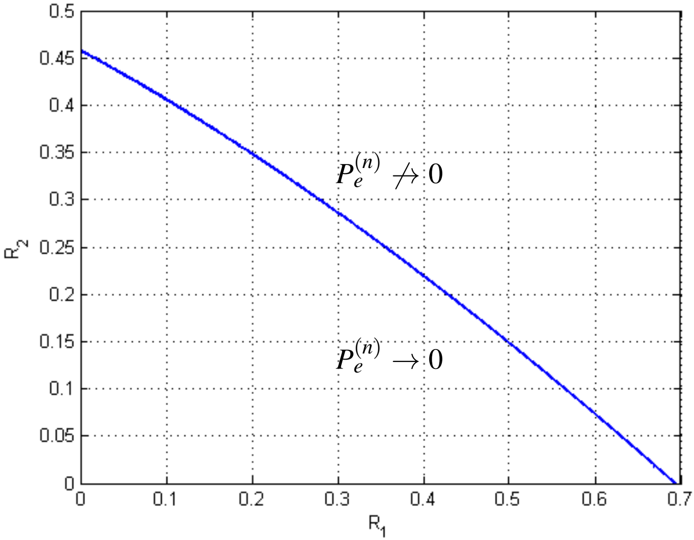

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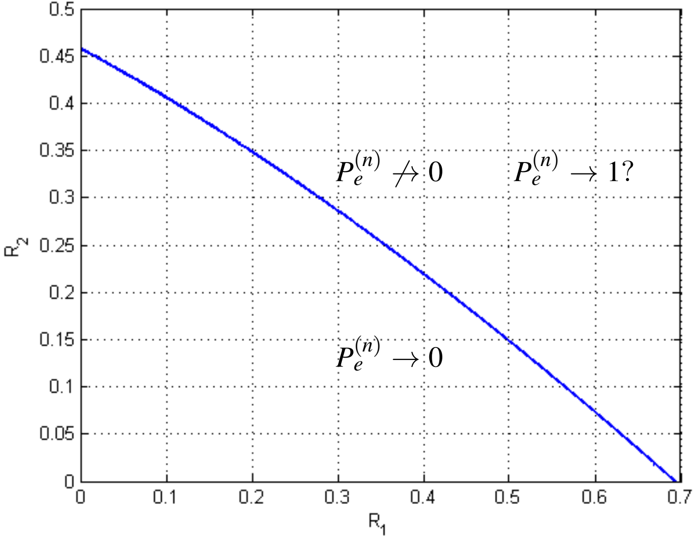

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  - BUL doesn't work for continuous alphabets [but see Wu and Özgür (2015)]
- Oohama (2015) uses properties of the Rényi divergence
  - Good bounds between the Rényi divergence $D_\alpha(P\|Q)$ and the relative entropy $D(P\|Q)$ exist for finite alphabets

ε-Capacity Region

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- Capacity region $\mathcal{C}_\varepsilon$ is the set of all achievable rate pairs
- Strong converse holds iff $\mathcal{C}_\varepsilon$ does not depend on ε.
- We already know that

$$\mathcal{R}_{\text{BC}} = \mathcal{C}_0 \subset \mathcal{C}_\varepsilon$$

Strong converse

Theorem

The Gaussian BC satisfies the strong converse property:

$$\mathcal{C}_\varepsilon = \mathcal{R}_{\mathrm{BC}}, \qquad \forall\, \varepsilon \in [0, 1)$$

Key ideas in proof:

- Derive an appropriate information spectrum converse bound
- Use the Gaussian Poincaré inequality to bound relevant variances

Weak Converse for GBC [Bergmans (1974)]

- Step 1: Invoke Fano's inequality to assert that for any sequence of codes with vanishing error probability $\varepsilon_n \to 0$,

$$R_j \le \frac{1}{n} I(W_j; Y_j^n) + o(1), \qquad \forall j \in \{1, 2\}.$$

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- Step 2: Single-letterize and entropy power inequality

$$I(W_1; Y_1^n) \leq nI(X; Y_1|U) \overset{EPI}{\leq} n\mathrm{C}\left(\frac{\alpha P}{\sigma_1^2}\right)$$

$$I(W_1; Y_1^n) + I(W_2; Y_2^n) \leq nI(U; Y_2) \overset{EPI}{\leq} n\mathrm{C}\left(\frac{(1-\alpha)P}{\alpha P + \sigma_2^2}\right)$$

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where $U_i := (W_2, Y_1^{i-1})$. One also uses the degradedness condition here:

$$I(W_2, Y_2^{i-1}, Y_1^{i-1}; Y_{2i}) = I(U_i; Y_{2i}).$$

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$$\varepsilon' \ge \Pr\left(\log \frac{P(Y_1^n | w_1)}{P(Y_1^n)} \le nR_1 - \gamma_1(w_1, w_2) \right) - n^2 e^{-\gamma_1(w_1, w_2)} - \mathbf{1}\left\{ 2^{n(R_1+R_2)} \int_{\mathcal{D}_1(w_1)} P(y_1^n) P(w_2 | w_1, y_1^n) \, dy_1^n > n^2 \right\}$$

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- Establish a bound on the coding rate

$$nR_1 \le \mathbb{E}\left[\log \frac{P(Y_1^n|w_1)}{P(Y_1^n)}\right] + \sqrt{\frac{2}{1-\varepsilon'} \operatorname{var}\left[\log \frac{P(Y_1^n|w_1)}{P(Y_1^n)}\right]} + 3\log n$$

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- Use the Gaussian Poincaré inequality with careful identification of f and peak power constraint $\|X^n\|^2 \le nP$ to assert that

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- Backoff term is of the order $O(1/\sqrt{n})$, i.e.,

$$\lambda \log M_{1n}^* + (1-\lambda)\log M_{2n}^* = n\mathrm{C}_\lambda + O(\sqrt{n})$$

where

$$\mathrm{C}_\lambda := \max_{\alpha\in[0,1]} \left\{ \lambda \mathrm{C}\Big(\frac{\alpha P}{\sigma_1^2}\Big) + (1-\lambda)\mathrm{C}\Big(\frac{(1-\alpha)P}{\alpha P + \sigma_2^2}\Big) \right\}$$

but nailing down the constant (dispersion) seems challenging.

Another Application: Quasi-Static Fading Channels

- Consider the channel model

$$Y_i = \sqrt{H} X_i + Z_i, \qquad i = 1, \ldots, n$$

where Z_i are independent standard normal random variables and

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- Delay-limited capacity [Hanly and Tse (1998)], i.e., the maximum transmission rate under the constraint that the maximal error probability over all $H > 0$ vanishes

Vanishing Normalized Relative Entropy

- The delay-limited capacity [Hanly and Tse (1998)] is

$$\mathrm{C}(P_{\mathrm{DL}}), \quad \text{where} \quad P_{\mathrm{DL}} := \frac{P}{\mathbb{E}[1/H]}$$

- For any sequence of capacity-achieving codes with vanishing maximum error probability

$$\lim_{n\to\infty} \frac{1}{n} D\big(P_{Y^n} \| (P_Y^*)^n\big) \to 0 \quad \text{where} \quad P_Y^* = \mathcal{N}(0, P_{\mathrm{DL}}).$$

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- Control a variance term

$$\mathrm{var}\left[\log \frac{P_{Y^n|X^n,H}(Y^n|X^n,h)}{P_{Y^n|H}(Y^n|h)}\right] = O(n).$$

Concluding Remarks

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- Allows us to establish strong converses and properties of good codes by controlling variance of log-likelihood ratios
- For more details, please see
  - Arxiv: 1509.01380 (Strong converse for Gaussian broadcast)
  - Arxiv: 1510.08544 (Empirical output distribution of good codes for fading channels)