Exact Error and Erasure Exponents for the Asymmetric Broadcast Channel

Matlab Code

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I. Parameters for the Matlab Code

- For the exponent $\mathcal{G}$, in the Matlab code, the distribution $\hat{Q}_{UXY}$ is denoted as "K0yuK0x_yu" and the conditional distribution $Q_{X|UY}$ is denoted as "K1x_yu". These distributions are generated from the corresponding parameter X according to

$$K0yu(*) = 0.5 * [X(1), 1 - X(1); 1 - X(1), X(1)] \tag{1}$$
$$K0x\_yu(0|*) = [X(2), X(3), 1 - X(3), 1 - X(2)] \tag{2}$$
$$K1x\_yu(0|*) = [X(4), X(5), 1 - X(5), 1 - X(4)] \tag{3}$$

  where the symbol "$*$" is denoted as $\{00, 01, 10, 11\}$.
- For the exponent $\mathcal{D}$, in the Matlab code, the distribution $\hat{Q}_{UXY}$ is denoted as "K0yuK0x_yu" and the conditional distribution Q_{UX_Y} is denoted as "Qu_yK1x_yu". These distributions are generated from the corresponding parameter X according to

$$K0yu(*) = 0.5 * [X(1), 1 - X(1), 1 - X(1), X(1)] \tag{4}$$
$$K0x\_yu(0|*) = [X(2)X(3)1 - X(3)1 - X(2)] \tag{5}$$
$$Qu\_y(0|*) = [X(4), 1 - X(4)] \tag{6}$$
$$K1x\_yu(0|*) = [X(5), X(6), 1 - X(6), 1 - X(5)] \tag{7}$$

  where the symbol "$*$" in (6) is denoted as $\{0, 1\}$.
- For the exponent $\mathcal{K}$, as the only difference between the exponents $\mathcal{K}$ and $\mathcal{G}$ is the constraints $\mathcal{L}_3$ and $\mathcal{L}_2$, respectively, for the inner optimization, i.e., the sub-optimization problem S_0 in $\mathcal{L}_3$. The setting in $\mathcal{K}$ is same to the one in $\mathcal{D}$, and the conditional distribution $\tilde{Q}_{X|UY}$ in the sub-optimization S_0 is denoted as "H1x_yu" in the Matlab code and is generated from the corresponding parameter X according to

$$H1x\_yu(0|*) = [X(1), X(2), 1 - X(2), 1 - X(1)] \tag{8}$$

The symmetry of these distributions above is due to the discussion in the follow section.

II. Simplification for the Symmetric Binary-case

In the symmetric binary-case, We will show the optimal (conditional) distributions satisfy some symmetry properties in the above section due to the KKT analysis.

For the exponent $\mathcal{G}$, we first consider the inner optimization problem, i.e. the distribution $\hat{Q}_{UXY}$ is fixed. Assume that $Q_{X|UY}(0|*) = \{q_{00}, q_{10}, q_{01}, q_{11}\}$. From the KKT analysis, we can get

$$q_{00} + q_{11} = 1 \quad q_{10} + q_{01} = 1 \tag{9}$$

where this property do not depend on the distribution $\hat{Q}_{UXY}$. Then, by updating the conditional distribution $Q_{X|UY}(0|*)$ with the property (9) above, the object function of the inner optimization problem (also appears in the constraint $\mathcal{L}_2$) and the function $\mathbb{E}_Q \ln \frac{1}{W_{\mathcal{Y}}}$ in the constraint $\mathcal{L}_2$ only (linearly) depend on the values $\hat{q}_1 \triangleq \hat{Q}_{UY}(0,0) + \hat{Q}_{UY}(1,1)$ and $\hat{q}_2 \triangleq \hat{Q}_{UY}(0,1) + \hat{Q}_{UY}(1,0)$. Therefore, the optimal value of the inner optimization problem only depends on the values of $(\hat{q}_1, \hat{q}_2)$ and $\hat{q}_0 \triangleq \mathbb{E}_{\hat{Q}} \ln \frac{1}{W_{\mathcal{Y}}}$. Then we can simplify the KKT analysis for the whole optimization problem $\mathcal{G}$ as follow:

$$\mathcal{G} = \min_{(\hat{q}_1, \hat{q}_2, \hat{q}_0)} \Big[\min_{\hat{Q}_{UXY}} D(\hat{Q}_{UXY} \| P_{UXY}) + \mu(\hat{q}_1, \hat{q}_2, \hat{q}_0) \Big] \tag{10}$$

where $\mu(\hat{q}_1, \hat{q}_2, \hat{q}_0)$ denotes the inner optimization of the original problem $\mathcal{G}$, and where the inner minimization is over

$$\hat{Q}_{UXY} : \begin{cases} \hat{Q}_{UY}(0,0) + \hat{Q}_{UY}(1,1) = \hat{q}_1 \\ \hat{Q}_{UY}(0,1) + \hat{Q}_{UY}(1,0) = \hat{q}_2 \\ \mathbb{E}_{\hat{Q}} \ln \dfrac{1}{W_{\mathcal{Y}}} = \hat{q}_0 \end{cases} \tag{11}$$

Then, by using the the KKT analysis for the inner optimization of (10), we can get the symmetry property for $\hat{Q}_{UXY}$

$$\hat{Q}_U(0) = \hat{Q}_U(1) = 0.5 \tag{12}$$

$$\hat{Q}_{Y|U}(1|0) = \hat{Q}_{Y|U}(0|1) \tag{13}$$

$$\hat{Q}_{X|UY}(0|00) + \hat{Q}_{X|UY}(0|11) = 1 \tag{14}$$

$$\hat{Q}_{X|UY}(0|01) + \hat{Q}_{X|UY}(0|10) = 1 \tag{15}$$

Therefore, the parameters for the optimization problem $\mathcal{G}$ can be reduced from 11 to 5. Similarly, we can also get the symmetry property for the optimization problem $\mathcal{D}$ and reduce the number of the parameters from 13 to 6.

For the remained optimization problem $\mathcal{K}$. As there exists a sub-minimization problem S_0 in the constraint $\mathcal{L}_3$, for deriving the symmetry properties for the (conditional) distributions $\hat{Q}_{UXY}$ and $\tilde{Q}_{X|UY}$, we can either use a similar method above to deal with the KKT analysis carefully or rewrite the optimization to clear about KKT analysis by moving the minimization (S_0) in the constraint $\mathcal{L}_3$ to the object function of $\mathcal{K}$ as

$$\mathcal{K} \triangleq \min_{\hat{Q}_{UXY}} \Big[D(\hat{Q}_{UXY} \| P_{UXY}) + \max_{\tilde{Q}_{X|UY}} \min_{Q_{UX|Y} \in \mathcal{L}_3'} \Psi(Q_{UX|Y}\hat{Q}_Y, R_1, R_2) \Big] \tag{16}$$

where the maximization is over $\tilde{Q}_{X|UY} : \beta(\tilde{Q}_{X|UY}\hat{Q}_{UY}, R_1) \le 0$, and where

$$\mathcal{L}_3' \triangleq \Big\{ Q_{UX|Y} : \mathbb{E}_Q \ln \frac{1}{W_{\mathcal{Y}}} - [\beta(\tilde{Q}, R_1) - \mathbb{E}_{\tilde{Q}} \ln W_{\mathcal{Y}}] - T \le \Delta(Q, R_1, R_2) \Big\} \tag{17}$$

with $Q = Q_{UX|Y}\hat{Q}_Y$ and $\tilde{Q} = \tilde{Q}_{X|UY}\hat{Q}_{UY}$. Therefore, we can also get the symmetry property for the optimization problem $\mathcal{K}$ and reduce the number of the parameters from 17 to 8.

As the codebook and the received sequence satisfies Markov property $U - X - Y$, we may hope the optimal distributions $\hat{Q}$, Q and $\tilde{Q}$ on $\mathcal{U} \times \mathcal{X} \times \mathcal{Y}$ also satisfy Markov property $U - X - Y$. Interestingly, if these distributions satisfy Markov property $U - X - Y$ and their margin distributions on $\mathcal{U} \times \mathcal{X}$ and $\mathcal{X} \times \mathcal{Y}$ are some binary symmetric distribution, it can be easily checked that these distributions satisfy the symmetry properties described above. However, we can NOT obtain Markov property $U - X - Y$ for these optimal distributions from the symmetry properties described above, and can also find that these optimal distributions do not satisfy Markov property $U - X - Y$ for our most cases.