Second-Order Asymptotics for the Gaussian MAC with Degraded Message Sets

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$$\log M^*(n, \varepsilon, S) = n\mathsf{C}(S) + \sqrt{n\mathsf{V}(S)}\Phi^{-1}(\varepsilon) + o(\sqrt{n}) \quad \text{nats},$$

where the Gaussian capacity and Gaussian dispersion functions are defined as

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$$\mathsf{C}(S) := \frac{1}{2}\log(1+S), \qquad \mathsf{V}(S) := \frac{S(S+2)}{2(S+1)^2}$$

- Second-order coding rate $=$ Largest coefficient of $\sqrt{n}$ term $= \sqrt{\mathsf{V}(S)}\Phi^{-1}(\varepsilon)$.

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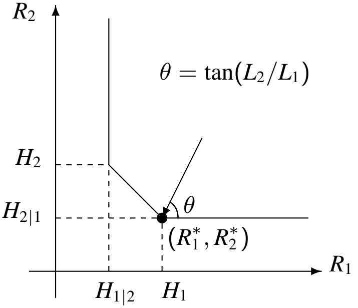

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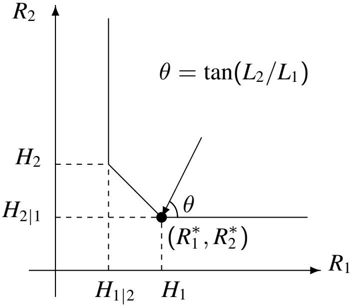

$$(L_1, L_2) \in \mathcal{L}(\varepsilon; R_1^*, R_2^*)$$

implies exists ε-reliable codes with

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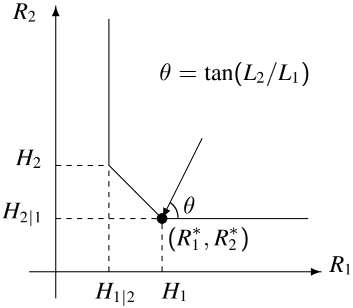

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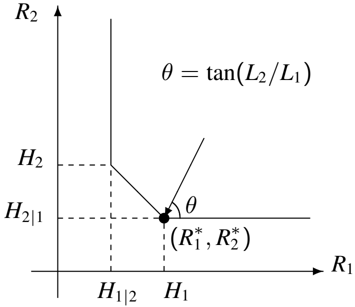
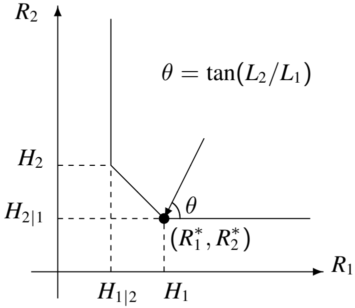


$$\mathcal{L}(\varepsilon; H_1, H_{2|1}) = \{(L_1, L_2) : \Psi(L_2, L_1 + L_2; \mathbf{V}_{2,12}) \geq 1 - \varepsilon\} .$$

where



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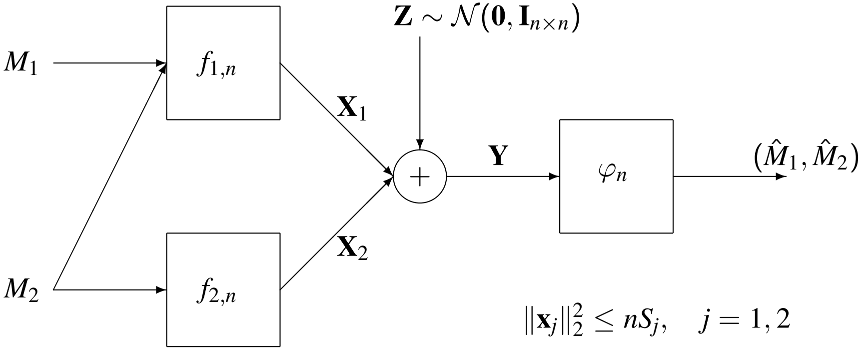
- We consider a MAC-like model that retains the main characteristics of MAC but skirts the problems above

Gaussian MAC with Degraded Message Sets



$$\|\mathbf{x}_j\|_2^2 \le nS_j, \quad j = 1, 2$$

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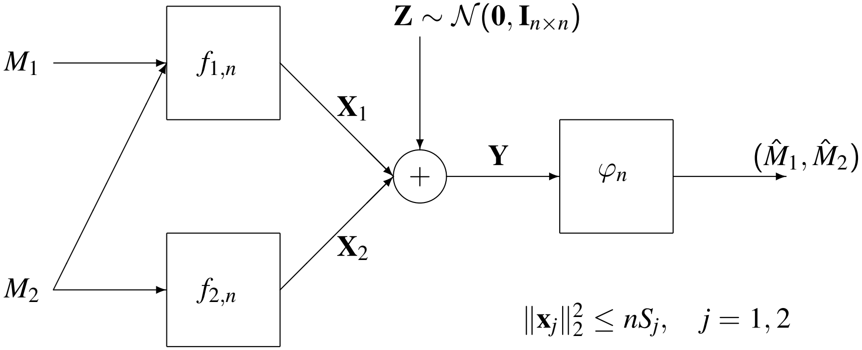

- Encoder 1 has access to both messages
- Capacity region is well known [Exercise 5.18(b), El Gamal and Kim (2012)]; achieved using superposition coding

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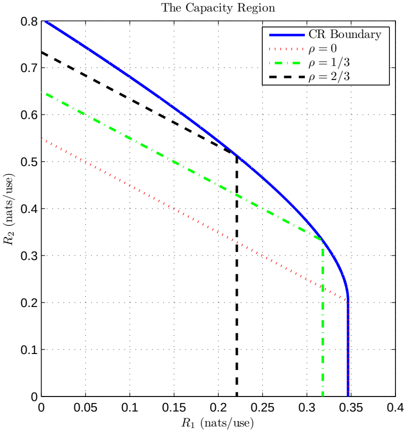

$$R_1 \leq \mathsf{C}\big((1-\rho^2)S_1\big)$$

$$R_1 + R_2 \leq \mathsf{C}\big(S_1 + S_2 + 2\rho\sqrt{S_1 S_2}\big)$$

$$\mathsf{C}(x) = \frac{1}{2}\log(1+x)$$

$\rho \in [0,1]$ parametrizes curved boundary and indicates the amount of correlation between users' codewords.

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- Main Contribution: A complete characterization of $\mathcal{L}(\varepsilon; R_1^*, R_2^*)$
- First complete characterization of second-order asymptotics for a channel-type network information theory problem

Some Basic Definitions

- Mutual informations

$$\mathbf{I}(\rho) := \begin{bmatrix} I_1(\rho) \\ I_{12}(\rho) \end{bmatrix} = \begin{bmatrix} \mathsf{C}\big((1-\rho^2)S_1\big) \\ \mathsf{C}\big(S_1 + S_2 + 2\rho\sqrt{S_1 S_2}\big) \end{bmatrix}$$

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- Dispersions $\mathsf{V}(x,y) := \frac{x(y+2)}{2(x+1)(y+1)}$ and $\mathsf{V}(x) := \mathsf{V}(x,x)$

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Generalization of Inverse CDF of a Gaussian

- For a positive semi-definite matrix $\mathbf{V}$,

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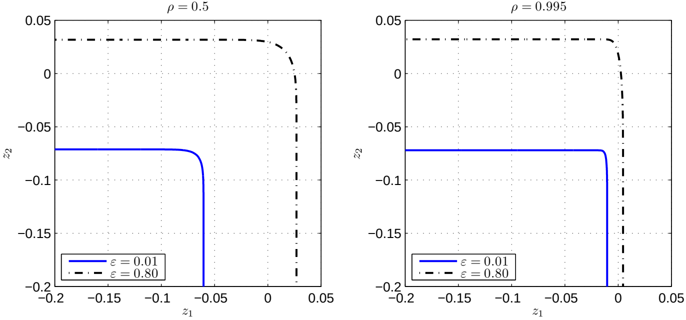

The Main Result: Vertical Boundary

Points on vertical boundary reduce to scalar dispersion as sum rate constraint is in error exponents regime [Haim-Erez-Kochman (2012)]

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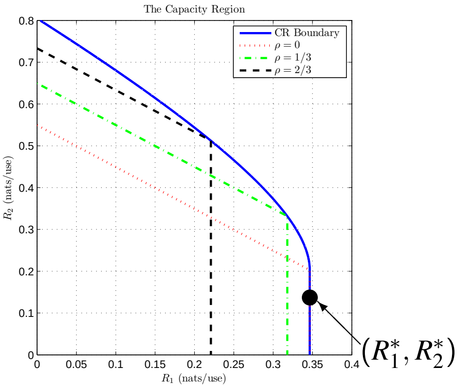

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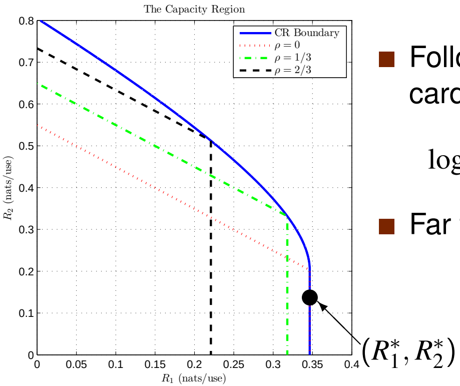

- Following expansion holds for cardinality of first codebook

$$\log M_{1,n} \approx nI_1(0) + \sqrt{nV_1(0)}\Phi^{-1}(\varepsilon)$$

- Far from sum rate constraint

$$\lim_{n\to\infty} \frac{1}{n} \log(M_{1,n}M_{2,n}) < I_{12}(0)$$

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Radically different behavior in the curved region

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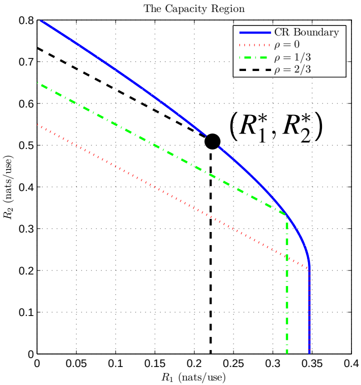

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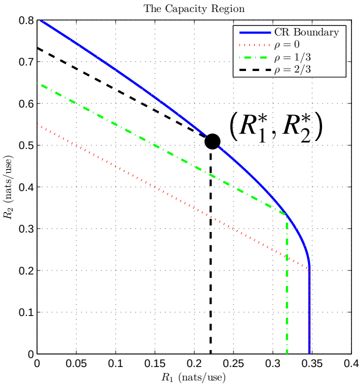

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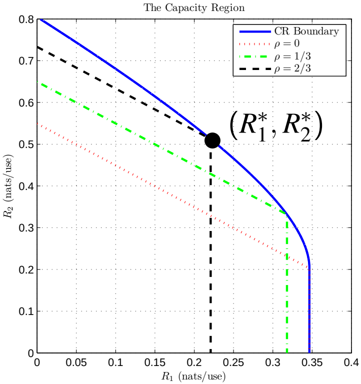

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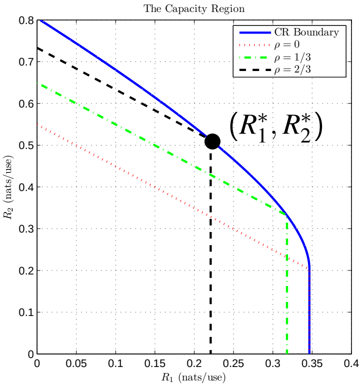

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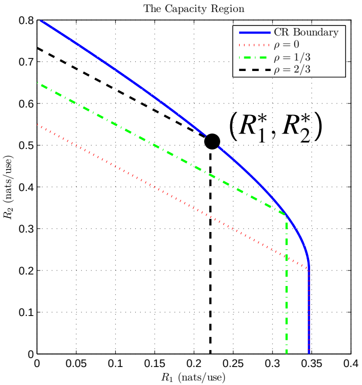

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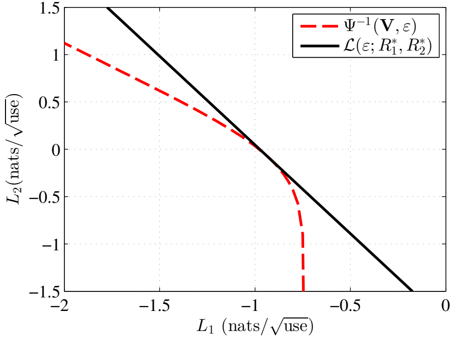
- Use $\mathcal{N}(\mathbf{0}, \mathbf{\Sigma}(\rho_n))$ and $\rho_n = \rho + \beta/\sqrt{n}$

Illustration of Second-Order Coding Rates



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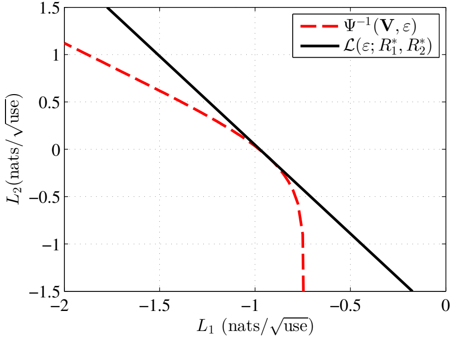

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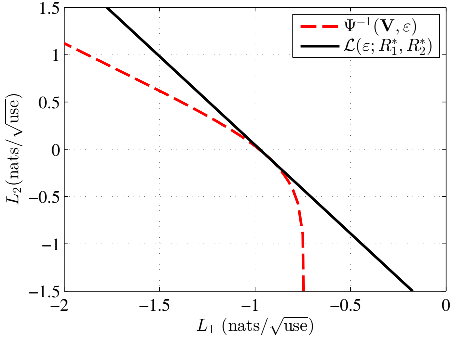

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- Second-order rates achieved using a single input distribution $\mathcal{N}(\mathbf{0}, \mathbf{\Sigma}(\rho))$ is not optimal
- The optimal second-order coding rate region is a half-space

Main Ideas in Converse Proof: Part I

- By a standard $n \to n+1$ argument [Shannon (1959)], we may consider codes with equal power constraints

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- Reduction to constant correlation type classes

$$\mathcal{T}_n(k) = \left\{ (\mathbf{x}_1, \mathbf{x}_2) : \frac{\langle \mathbf{x}_1, \mathbf{x}_2 \rangle}{\|\mathbf{x}_1\|_2 \|\mathbf{x}_2\|_2} \in \left(\frac{k-1}{n}, \frac{k}{n} \right] \right\}, \quad k = 1, 2, \ldots, n.$$

  Without too much loss in rate

Main Ideas in Converse Proof: Part II

- Verdú-Han-type converse: For any $\gamma > 0$ and any $(Q_{\mathbf{Y}|\mathbf{X}_2}, Q_{\mathbf{Y}})$, have the following non-asymptotic converse bound

$$\varepsilon_n \geq \Pr\Bigg(j_1(\mathbf{X}_1, \mathbf{X}_2, \mathbf{Y}) \leq \frac{1}{n}\log M_{1,n} - \gamma \quad \text{or} \quad j_{12}(\mathbf{X}_1, \mathbf{X}_2, \mathbf{Y}) \leq \frac{1}{n}\log(M_{1,n}M_{2,n}) - \gamma \Bigg) - 2e^{-n\gamma}$$

where $j_1(\mathbf{x}_1, \mathbf{x}_2, \mathbf{y}) = \frac{1}{n}\log\frac{W^n(\mathbf{y}|\mathbf{x}_1,\mathbf{x}_2)}{Q_{\mathbf{Y}|\mathbf{X}_2}(\mathbf{y}|\mathbf{x}_2)}$ and $j_{12}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{y}) = \frac{1}{n}\log\frac{W^n(\mathbf{y}|\mathbf{x}_1,\mathbf{x}_2)}{Q_{\mathbf{Y}}(\mathbf{y})}$.

Main Ideas in Converse Proof: Part II

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- Let $\mathbf{j} = [j_1, j_{12}]^T$. Choose $(Q_{\mathbf{Y}|\mathbf{X}_2}, Q_{\mathbf{Y}})$ and for $(\mathbf{x}_1, \mathbf{x}_2) \in \mathcal{T}_n(k)$,

$$\mathbb{E}\big[\mathbf{j}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{Y})\big] \approx \mathbf{I}(\rho) \quad \text{and} \quad \mathrm{Cov}\big[\sqrt{n}\,\mathbf{j}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{Y})\big] \approx \mathbf{V}(\rho)$$

where

$$\frac{k-1}{n} \leq \rho \leq \frac{k}{n}$$

Main Ideas in Converse Proof: Part III

- By evaluating Verdú-Han using multivariate Berry-Esseen, there exists a sequence $\{\rho_n\}_{n\geq 1} \subset [0,1]$ satisfying

$$\rho_n = \rho + O\left(\frac{1}{\sqrt{n}}\right)$$

such that

$$\frac{1}{n}\begin{bmatrix} \log M_{1,n} \\ \log(M_{1,n}M_{2,n}) \end{bmatrix} \in \mathbf{I}(\rho_n) + \frac{\Psi^{-1}(\mathbf{V}(\rho_n),\varepsilon)}{\sqrt{n}} + o\left(\frac{1}{\sqrt{n}}\right)\mathbf{1}$$

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- By a Taylor expansion of $\mathbf{I}(\rho_n)$ around $\mathbf{I}(\rho)$

$$\mathbf{I}(\rho_n) \approx \mathbf{I}(\rho) + (\rho_n - \rho)\mathbf{D}(\rho)$$

and the Bolzano-Weierstrass theorem, we establish the converse for a point on the curved boundary

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- Full version: `http://arxiv.org/abs/1310.1197`