Scaling Laws for Learning High-Dimensional Markov Forest Distributions

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Allerton Conference (Sep 29, 2010)

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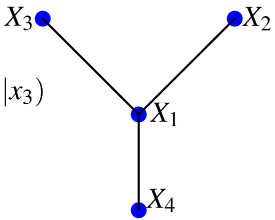

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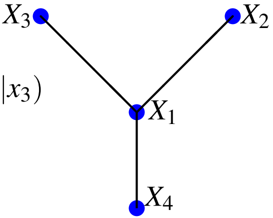

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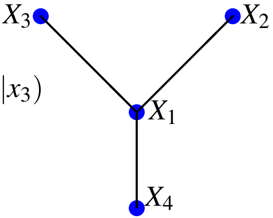

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- What if we want a larger class of acyclic models?

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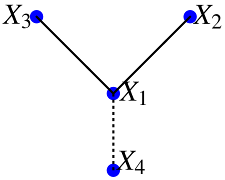

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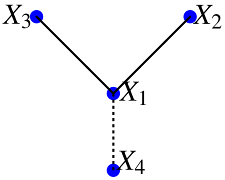

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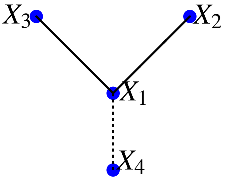

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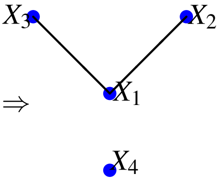

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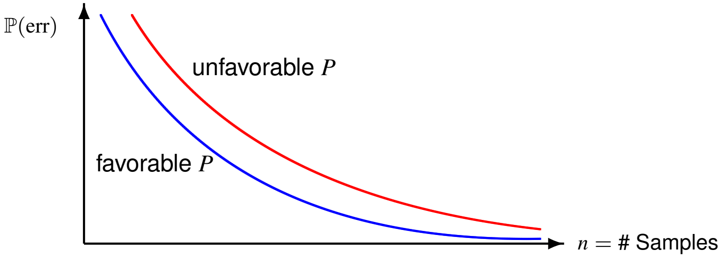

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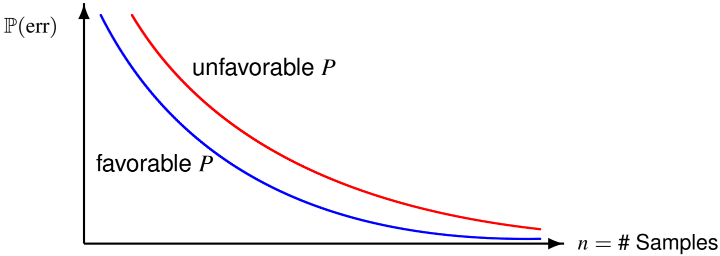

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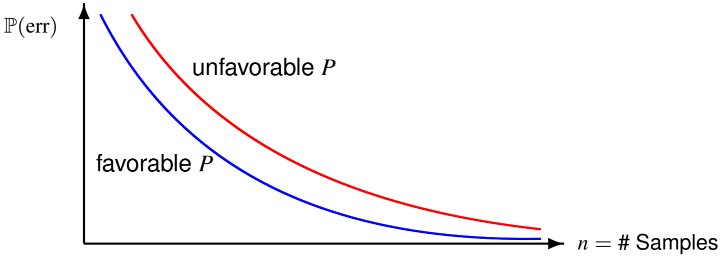

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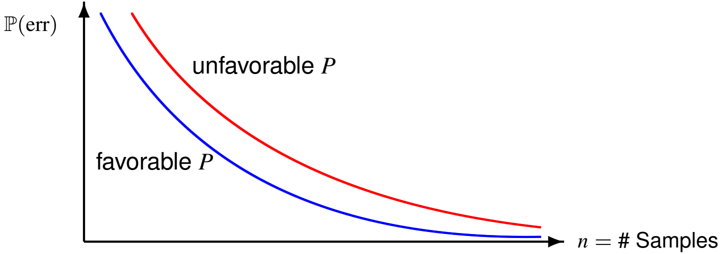

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  1. Number of samples n
  2. Number of variables d
  3. Number of edges $k \leq d - 1$

Main Contributions

- Propose CLThres, a thresholding algorithm, for consistently learning forest-structured models.

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- Prove convergence rates ("moderate deviations") for a fixed discrete graphical model $P \in \mathcal{P}(\mathcal{X}^d)$.
- Prove achievable scaling laws on (n, d, k) for consistent recovery in high-dimensions. Roughly speaking,

$$n > C_1 \log^{1+\delta}(d-k), \qquad \forall \delta > 0$$

is achievable.

Problem Setup

- Let $\mathcal{X}$ be a finite set and let $\mathcal{P}(\mathcal{X}^d)$ be the probability simplex over $\mathcal{X}^d$.
- We say that $P \in \mathcal{P}(\mathcal{X}^d)$ is a forest-structured model if it factorizes as

$$P(\mathbf{x}) = \prod_{i \in V} P(x_i) \prod_{(i,j) \in E_P} \frac{P(x_i, x_j)}{P(x_i)P(x_j)}$$

where $V = [1 : d]$ and $E_P \subset \binom{V}{2}$ and note $|E_P| \le d - 1$.

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- Given n i.i.d. samples $\{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$ drawn from P, a forest-structured model with edge set E_P.

- Output an estimate of the structure $\widehat{E}$.

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- How to deal with classic tradeoff between over- and underestimation errors?

The CLThres Algorithm

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- Output the forest with the top $\widehat{k}_n$ edges.
- Computational Complexity = $O((n + \log d)d^2)$.

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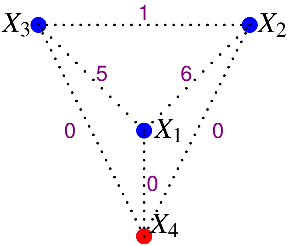

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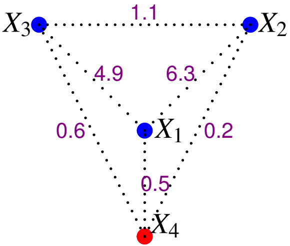

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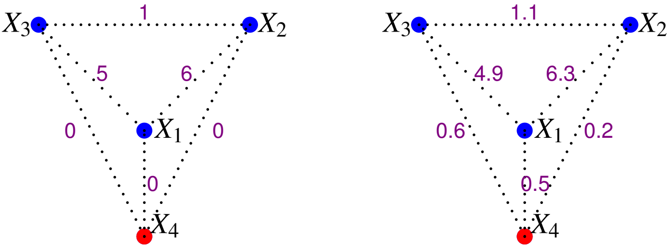

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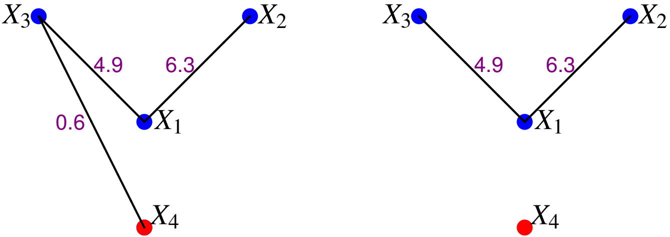

Max-weight spanning tree $\widehat{E}_{d-1}$ Thresholded Forest $\widehat{E}_{\widehat{k}_n}$

A Convergence Result for CLThres

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Roughly speaking, $\mathbb{P}(\widehat{E}_{\widehat{k}_n} \neq E_P) \approx \exp(-n\epsilon_n)$

Also have a "liminf" lower bound.

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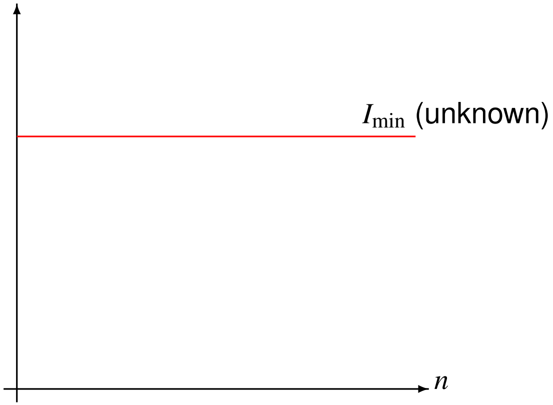
- Note that for two independent random variables X_i and X_j with product pmf $Q_i \times Q_j$,

$$\mathrm{std}(\widehat{I}(X_i; X_j)) = \Theta(1/n)$$

  Since the sequence $\epsilon_n = \omega(\log n/n)$ decays slower than $\mathrm{std}(\widehat{I}(X_i; X_j))$, no overestimation asymptotically.

Pruning Away Weak Empirical Mutual Informations



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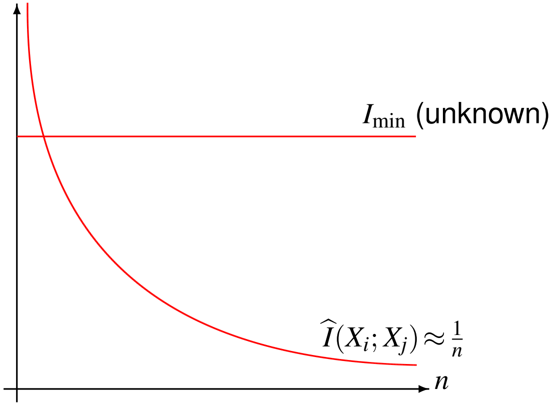

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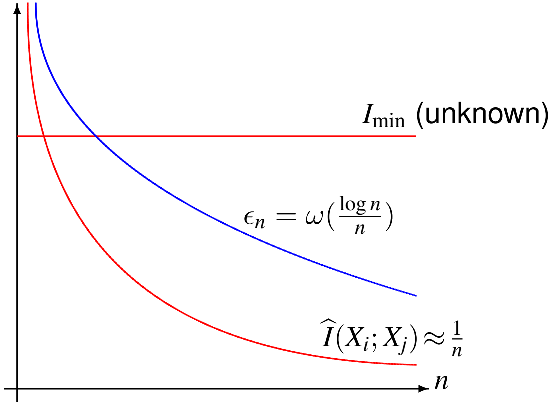

Asymptotically, ϵ_n will be smaller than $I_{\min}$ and larger than $\widehat{I}(X_i; X_j)$ with high probability.

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  Additional Technique: Ideas from Euclidean Information Theory [Borade and Zheng 2008].

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Assumptions

$$\text{(A1)} \qquad I_{\inf} := \inf_{d \in \mathbb{N}} \min_{(i,j) \in E_P} I(P_{i,j}) > 0$$

$$\text{(A2)} \qquad \kappa := \inf_{d \in \mathbb{N}} \min_{(x_i, x_j) \in \mathcal{X}^2} P_{i,j}(x_i, x_j) > 0$$

An Achievable Scaling Law for CLThres

Theorem ("Achievability")

Assume (A1) and (A2). Fix $\delta > 0$. Then if

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Theorem ("Achievability")

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the error probability of structure learning

$$\mathbb{P}(\mathit{error}) \to 0$$

as $(n, d_n, k_n) \to \infty$.

Remarks on the Achievable Scaling Law for CLThres

- If the model parameters (n, d, k) grow with n but if

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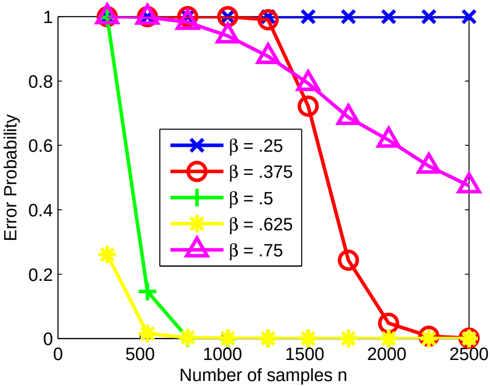
- Proof uses:
  1. Previous fixed d result.
  2. Exponents in the limsup upper bound do not vanish with increasing problem size as $(n, d_n, k_n) \to \infty$.

Finite Number of Samples?

There exists a tradeoff between under- and overestimation in the finite-sample case:



Design of

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takes into account the tradeoff.

But asymptotically, overestimation error dominates.

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Extensions:

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Extensions:

- Risk consistency has also been analyzed. See manuscript on arXiv.

Concluding Remarks

- Proposed a simple extension of Chow-Liu's max-weight spanning tree algorithm to learn forests consistently.
- Derived precise error rates in the form of a "moderate deviations" result.
- Derived scaling laws on (n, d, k) for structural consistency in high dimensions.

Extensions:

- Risk consistency has also been analyzed. See manuscript on arXiv.
- Need to find the right balance between over- and underestimation for the finite sample case.