Erasure and Undetected Error Probabilities in the Moderate Deviations Regime

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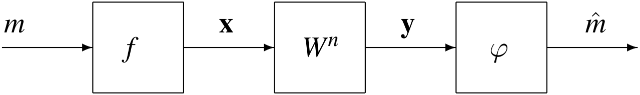
Channel Coding with the Erasure Option



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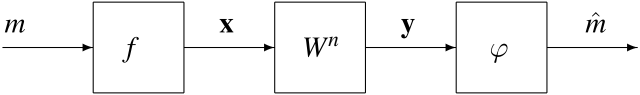

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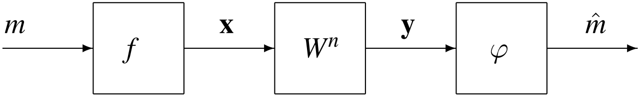

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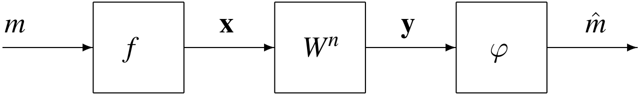

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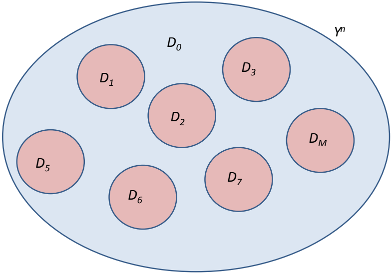
- If $\mathbf{y} \in \mathcal{D}_m$, declare m was sent; if $\mathbf{y} \in \mathcal{D}_0$, declare an erasure
- We assume W^n is a DMC, i.e.,

$$W^n(\mathbf{y} \,|\, \mathbf{x}) = \prod_{i=1}^{n} W(y_i \,|\, x_i)$$

Channel Coding with the Erasure Option : Illustration



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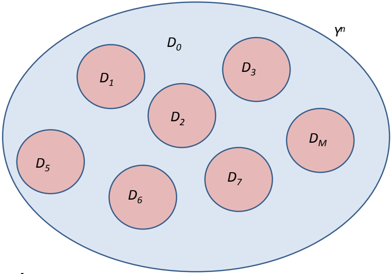

- Probability of total error

$$\Pr(\mathcal{E}_1 \,|\, \mathcal{C}_n) = \frac{1}{M_n} \sum_{m=1}^{M_n} \sum_{\mathbf{y} \in \mathcal{D}_m^c} W^n(\mathbf{y} \,|\, f_n(m)).$$

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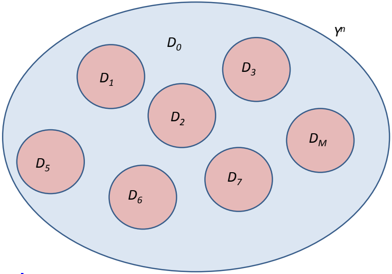

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- Probability of undetected error

$$\Pr(\mathcal{E}_2 \,|\, \mathcal{C}_n) = \frac{1}{M_n} \sum_{m=1}^{M_n} \sum_{\mathbf{y} \in \mathcal{D}_m} \sum_{m' \neq m} W^n(\mathbf{y} \,|\, f_n(m')).$$

Some Historical Remarks on Erasure Decoding

- Forney (1968) derived exponential error bounds for erasure and list decoding using Gallager-type bounding techniques

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- Somekh-Baruch and Merhav (2011) showed that the the type class enumerator method is ensemble-tight
- Tan-Moulin (2014) considered non-vanishing total and undetected error version of this problem.

Various Asymptotic Regimes

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- When $0 < t < 1/2$, this is the moderate deviations regime and

$$\epsilon_n^* \approx \exp\left(-n^{1-2t}\frac{a^2}{2V}\right), \qquad \text{sub-exponential}$$

Altuğ-Wagner (2014) and Polyanskiy-Verdú (2010)

Our Contribution : Two Asymptotic Regimes

- Moderate Deviations Regime with $0 < t < 1/2$:

$$\Pr(\mathcal{E}_1 \,|\, \mathcal{C}_n) \approx \exp\left(-\Theta(n^{1-2t})\right), \qquad \Pr(\mathcal{E}_2 \,|\, \mathcal{C}_n) \approx \exp\left(-\Theta(n^{1-t})\right)$$

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- Similar to Somekh-Baruch and Merhav (2011), we seek ensemble-tight results, i.e., we seek asymptotic equalities for

$$\mathbb{E}_{\mathcal{C}_n}[\Pr(\mathcal{E}_1 \,|\, \mathcal{C}_n)], \quad \text{and} \quad \mathbb{E}_{\mathcal{C}_n}[\Pr(\mathcal{E}_2 \,|\, \mathcal{C}_n)]$$

where $\mathbb{E}_{\mathcal{C}_n}[\cdot]$ denotes expectation over a random codebook.

Preliminary Quantities

- Mutual information

$$I(P,W) = \sum_x P(x) \sum_y W(y|x) \log \frac{W(y|x)}{PW(y)}$$

- Conditional information variance

$$V(P,W) = \sum_x P(x) \sum_y W(y|x) \left[\log \frac{W(y|x)}{PW(y)} - D(W(\cdot|x) \| PW) \right]^2$$

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- Always assume $V_{\min}(W) > 0$.

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Theorem (Moderate Deviations Direct)

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Decoding regions are reminiscent of information spectrum analysis:

$$\tilde{\mathcal{D}}_m := \left\{ \mathbf{y} : \log \frac{W^n(\mathbf{y}|\mathbf{x}_m)}{(PW)^n(\mathbf{y})} \geq \log M_n + bn^{1-t} \right\}$$

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Total $\Pr(\mathcal{E}_1 \,|\, \mathcal{C}_n) \approx \exp\left(-n^{1-2t} \frac{(a-b)^2}{2V_{\min}(W)} \right)$

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and

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- Also need a stronger decoder; c.f. Forney (1968)

$$\mathcal{D}_m := \left\{ \mathbf{y} : \frac{W^n(\mathbf{y}|\mathbf{x}_m)}{\sum_{m' \neq m} W^n(\mathbf{y}|\mathbf{x}_{m'})} \ge \exp(nT_n) \right\}$$

T_n : Blocklength-varying threshold depending on regime.

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$$\lim_{n\to\infty} -\frac{1}{n^{1-t}} \log \mathbb{E}_{\mathcal{C}_n}[\Pr(\mathcal{E}_2 \,|\, \mathcal{C}_n)] = b$$

Remarks on Ensemble Tight Result

- Similar result for mixed regime, i.e.,

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- As b increases towards a, decoding regions become smaller; erasure probability higher; undetected probability smaller

Main Analysis Technique

- Recall Forney's optimum decoding rule

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$$F_n := \log \left(\sum_{m' \neq 1} W^n(Y^n|X^n_{m'}) \right) - \log W^n(Y^n|X^n_1)$$

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- Suffices to understand the cumulant generating function

$$\phi_n(s) := \log \mathbb{E}\left[\exp(sF_n)\right]$$

Here s doesn't have to be a constant; can vary with n

Cumulant Generating Function Asymptotics

Lemma

Consider $\phi_n(s) := \log \mathbb{E}\left[\exp(sF_n)\right]$. *Let* $0 < t \le 1/2$. *We have*

$$\phi_n\left(\frac{u}{n^t}\right) = \left(-au + u^2 \frac{V(P)}{2}\right) n^{1-2t} + o(1)$$

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- Concentration bounds

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- For example, if $t = 1/2$, we have

$$\lim_{n\to\infty} \phi_n\left(\frac{u}{\sqrt{n}}\right) = -au + u^2 \frac{V(P)}{2} \quad \Rightarrow \quad \text{Gaussian}$$

Modified Gärtner-Ellis Theorem

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Conclusion and Further Work

- We analyzed the total and undetected errors whose scalings are

$$\Pr(\mathcal{E}_1 \,|\, \mathcal{C}_n) \approx \exp\left(-\Theta(n^{1-2t})\right), \qquad \Pr(\mathcal{E}_2 \,|\, \mathcal{C}_n) \approx \exp\left(-\Theta(n^{1-t})\right)$$

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- Full version: `http://arxiv.org/abs/1407.0142`.