Equivocations, Exponents and Second-Order Rates under Various Rényi Information Measures

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IMS Workshop on Mathematics of IT Cryptography

The Main Question We Ask

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- Source A^n is correlated to E^n; their joint distribution is

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- Information-theoretic security: After application of f_n, how independent is A^n from E^n and how uniform is it for a given rate

$$R = \frac{1}{n} \log \|f_n\|$$

Measuring Independence and Uniformity

- We can measure dependence using the mutual information

$$I(A \wedge E) = D(P_{AE} \| P_A \times P_E).$$

We take $A \equiv f(A^n)$ and $E \equiv E^n$ later.

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- These are Shannon information measures

Rényi Information Measures

- The Rényi divergence of order $1+s$ is

$$D_{1+s}(P\|Q) := \frac{1}{s} \log \sum_a P(a)^{1+s} Q(a)^{-s}$$

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- Conditional Rényi entropy 1:

$$H_{1+s}(A|E|P_{AE}\|Q_E) := -D_{1+s}(P_{AE}\|I_A \times Q_E)$$
$$H_{1+s}(A|E) := H_{1+s}(A|E|P_{AE}\|P_E)$$

- Conditional Rényi entropy 2 (Gallager form):

$$H_{1+s}^{\uparrow}(A|E) := -\frac{1+s}{s}\log\sum_{e}\left(\sum_{a}P_{AE}(a,e)^{1+s}\right)^{\frac{1}{1+s}}$$

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$$\phi(s) := \log\sum_e\left(\sum_a P_{AE}(a,e)^{\frac{1}{1-s}}\right)^{1-s}$$

See Fehr and Berens (2014) for other definitions of conditional Rényi entropies.

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- Easy to check that

$$\max_{Q_E \in \mathcal{P}(\mathcal{E})} H_{1+s}(A|E|P_{AE}\|Q_E) = H^{\uparrow}_{1+s}(A|E).$$

Generalized Security Rényi Information Measures

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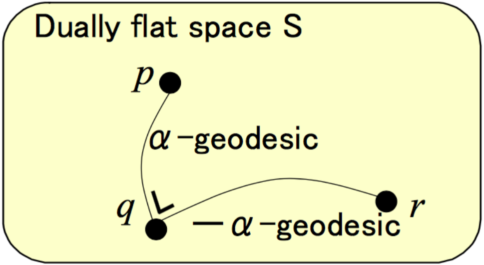
- When $s = 0$, these reduce to the information security measure based on relative entropy

$$C_1(A|E) = D(P_{AE} \| P_{\mathrm{mix},\mathcal{A}} \times P_E) = \log|\mathcal{A}| - H(A|E)$$

Remarks on Generalized Security Criteria I

Can show that

$$C_{1+s}(A|E) = I_{1+s}^{(\mathrm{Sibson})}(E \wedge A) + D_{1+s}(Q_A^{(s)} \| P_{\mathrm{mix},\mathcal{A}})$$



Generalized Pythagorean theorem in information geometry for Rényi entropy. Fig. from S. Akaho.

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- We can also show that

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- If $C_1(A|E)$ is small A is approximately independent of E and A is close to uniform.

Motivations for Generalized Security Measures

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  and min-entropy

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  are important; cf. leftover hash lemma

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- Error exponents for information-theoretic security problems

Families of Hash Functions

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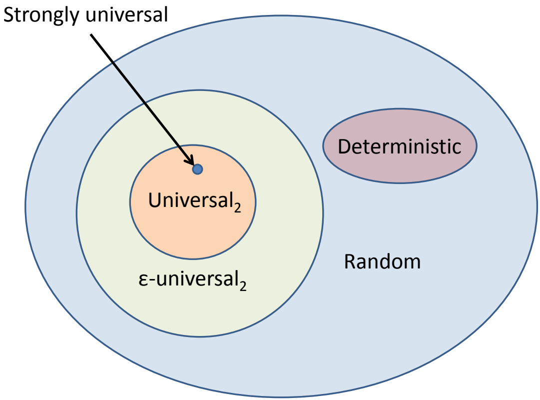
Definition

A random hash function is called strongly universal when the random variables $\{f_X(a) : a \in \mathcal{A}\}$ are independent and

$$\Pr(f_X(a) = m) = M^{-1}, \quad \forall\, a \in \mathcal{A}, m \in \mathcal{M}.$$

Families of Hash Functions



Hierarchy of hash functions.

Sequences of Hash Functions

Sequence of hash functions $\{f_{X_n} : \mathcal{A}^n \to [e^{nR}]\}_{n=1}^{\infty}$

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$\{X_n\}$: Sequence of common randomness independent of (A^n, E^n)

Asymptotics of Equivocation I

Define the averages of the security indices (over X_n) as

$$C_{1+s} := C_{1+s}(f_{X_n}(A^n)|E^n, X_n)$$
$$C^{\uparrow}_{1+s} := C^{\uparrow}_{1+s}(f_{X_n}(A^n)|E^n, X_n)$$

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Theorem

Let $M_n = \|f_{X_n}\| = \lfloor e^{nR} \rfloor$. For any $s \in [0,1]$, we have

$$\lim_{n\to\infty} \frac{1}{n} \inf_{f_{X_n}} C_{1+s} = |R - H_{1+s}(A|E)|^+,$$
$$\lim_{n\to\infty} \frac{1}{n} \inf_{f_{X_n}} C^{\uparrow}_{1+s} = |R - H^{\uparrow}_{1+s}(A|E)|^+.$$

Infima are over all random hash functions and achieved by any sequence of ϵ-almost universal$_2$ hash functions.

Asymptotics of Equivocation II

Theorem

Let $M_n = \|f_{X_n}\| = \lfloor e^{nR} \rfloor$*. For any* $s \in (0,1]$*, we have*

$$\lim_{n\to\infty} \frac{1}{n} \inf_{f_{X_n}} C_{1-s} = \begin{cases} R - H_{1-s}(A|E) & R \geq \hat{R}_{-s} \\ \max_{t\in[0,s]} \frac{t}{s}(R - H_{1-t}(A|E)) & R \leq \hat{R}_{-s} \end{cases} .$$

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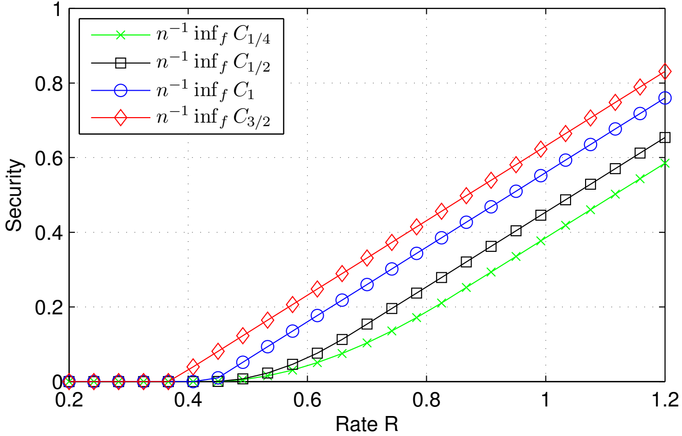
- Similar behavior for $C_{1-s}^{\uparrow}$

Asymptotics of Equivocation : Illustration



Security measures $\frac{1}{n}C_{1+s}$ and $\frac{1}{n}C_{1-s}$ for source $\begin{bmatrix} 0.7 & 0.1 \\ 0.1 & 0.1 \end{bmatrix}$

Optimal Key Generation Rate

Corollary

We have

$$\sup\left\{R : \lim_{n\to\infty} \inf_{f:\mathcal{A}^n\to[e^{nR}]} \frac{C_{1+s}}{n} = 0\right\} = \begin{cases} H_{1+s}(A|E) & \textit{if } s \in (0,1] \\ H(A|E) & \textit{if } s \in [-1,0] \end{cases}$$

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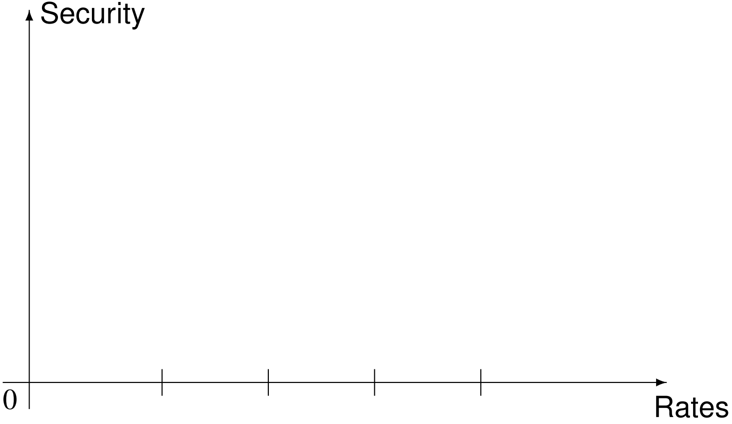
Summary of Behavior of Equivocations

Schematic showing the relation between the various entropies and the transition rate $\hat{R}_{-s}$.



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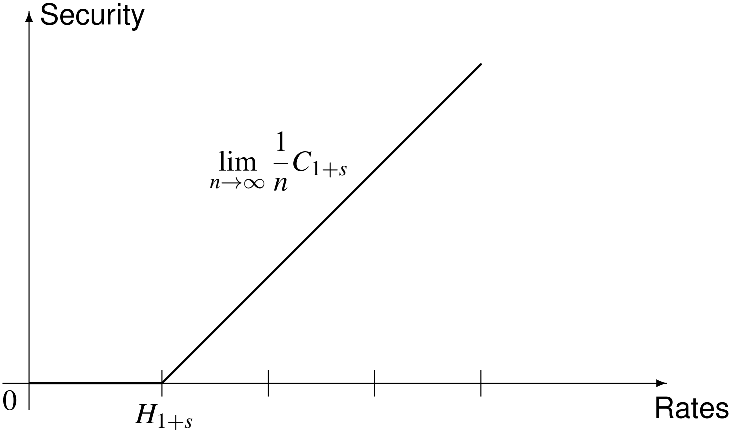

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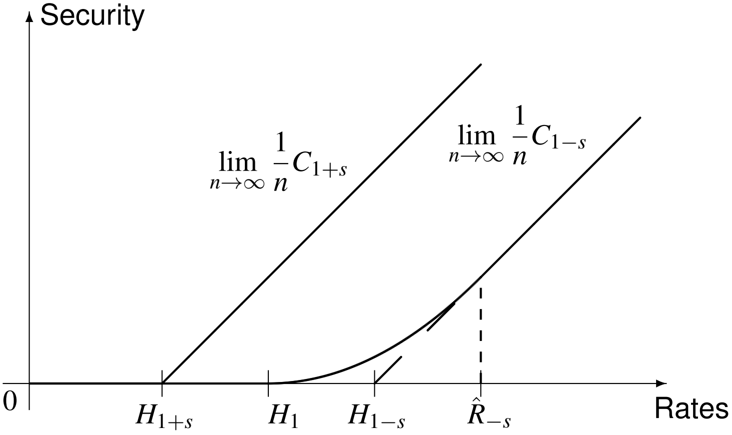

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- How fast do they tend to zero, i.e., we seek limiting behavior of the exponent

$$-\frac{1}{n} \log C_{1+s}(f_{X_n}(A^n)|E^n, X_n)$$

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Similar behavior for Gallager forms.

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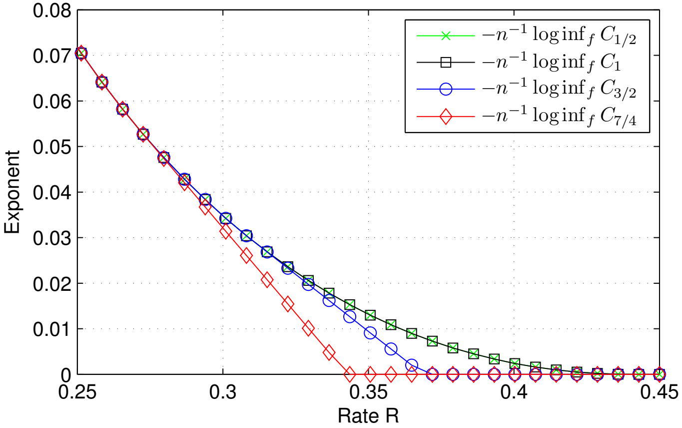
Illustration of the exponential decays of $C_{1/2}, C_1, C_{3/2}$, and $C_{7/4}$. Also indicated are the zero-crossings.



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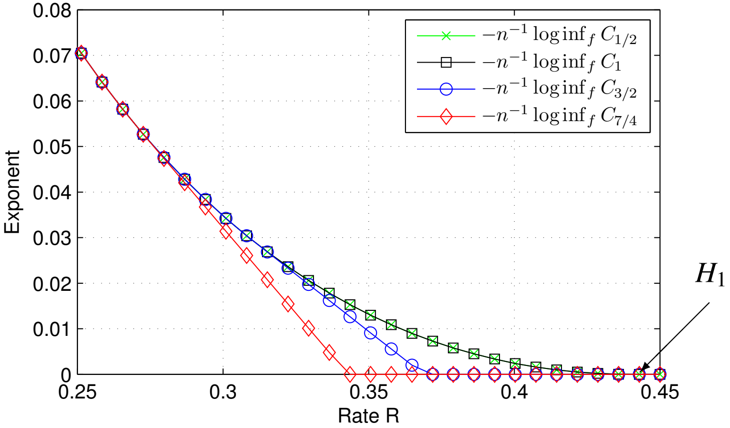

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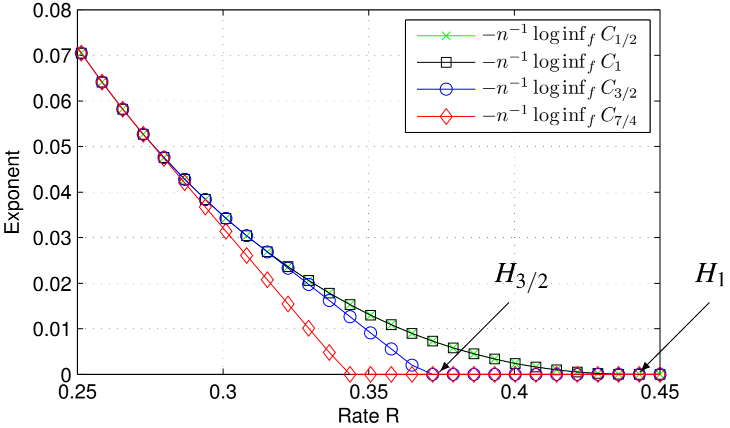

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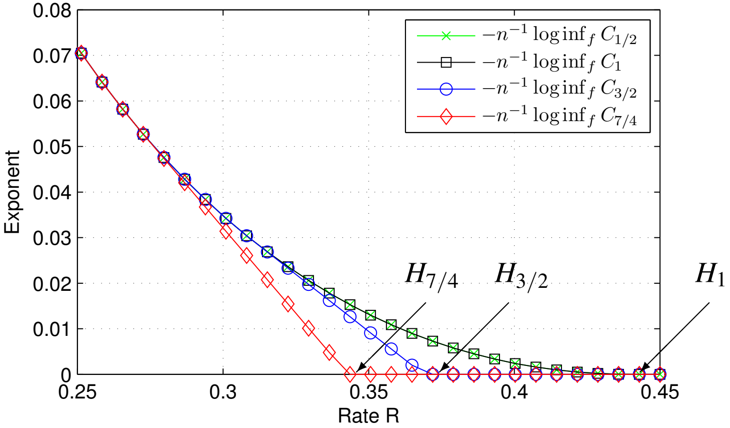

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- Non-asymptotic bounds are improved versions of Bennett et al.'s (1999) bounds for the leftover hash lemma stated in terms of the Rényi entropy of order 2

Second-Order Asymptotics

Now we assume that the key size M_n satisfies

$$\log M_n = nH(A|E) + \sqrt{n}L$$

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Here, we make use of central limit theorem ideas.

Second-Order Asymptotics: Illustration

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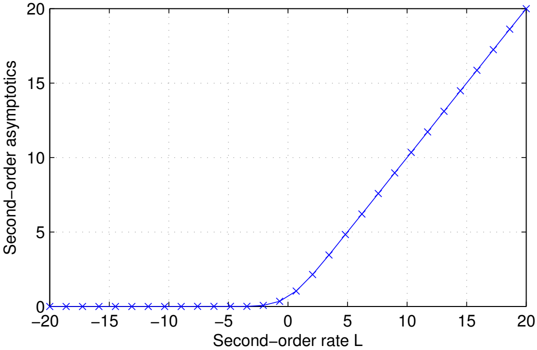

Summary

- We have conducted a detailed study of the asymptotic behavior of

$$C_{1+s} := \log M_n - H_{1+s}(f_{X_n}(A^n)|E^n, X_n)$$
$$C^{\uparrow}_{1+s} := \log M_n - H^{\uparrow}_{1+s}(f_{X_n}(A^n)|E^n, X_n)$$

their exponents, and second-order asymptotics.

$$\begin{tikzcd}
A^n \arrow[r, dash] \arrow[d, "f_{X_n}"'] & E^n \arrow[d] & X_n \arrow[d] \\
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- Optimal key generation rate has a surprising behavior which depends on the sign of s.
- For more results, consult the full version: `arxiv.org/abs/1504.02536` (IT Transactions revised)