Time-Division Transmission is Optimal for Covert Communication over Some Broadcast Channels

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ITW, Guangzhou (Nov 2018)

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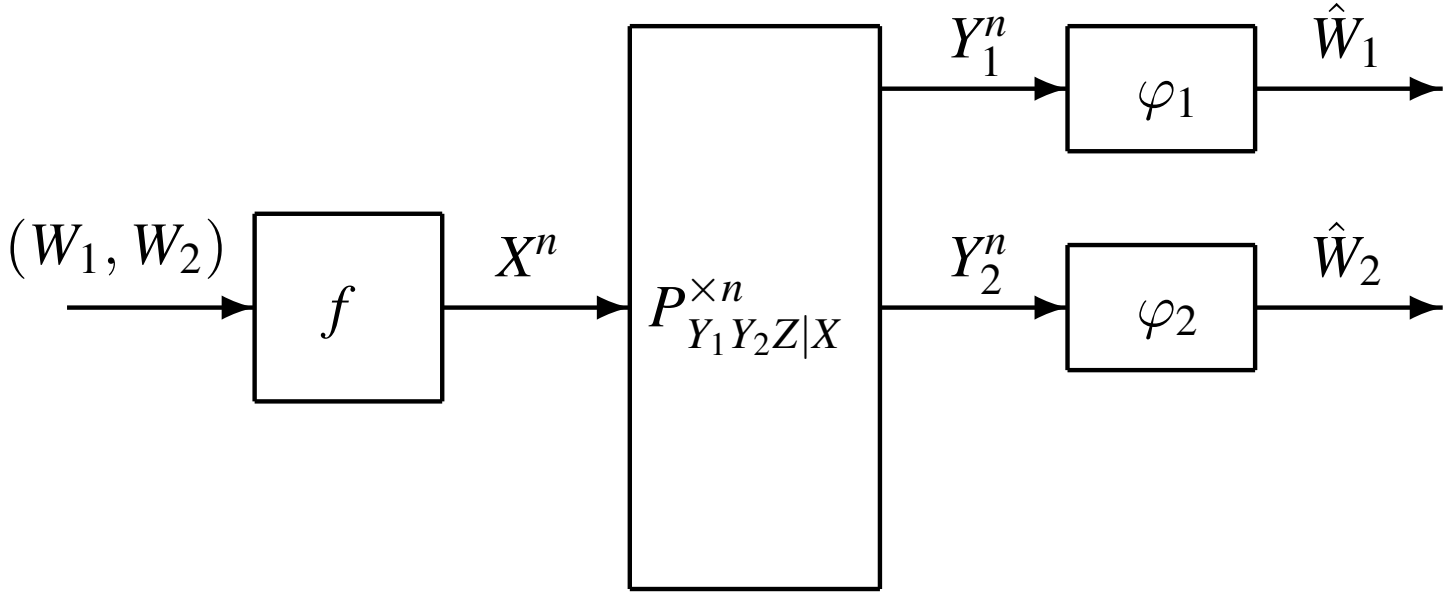

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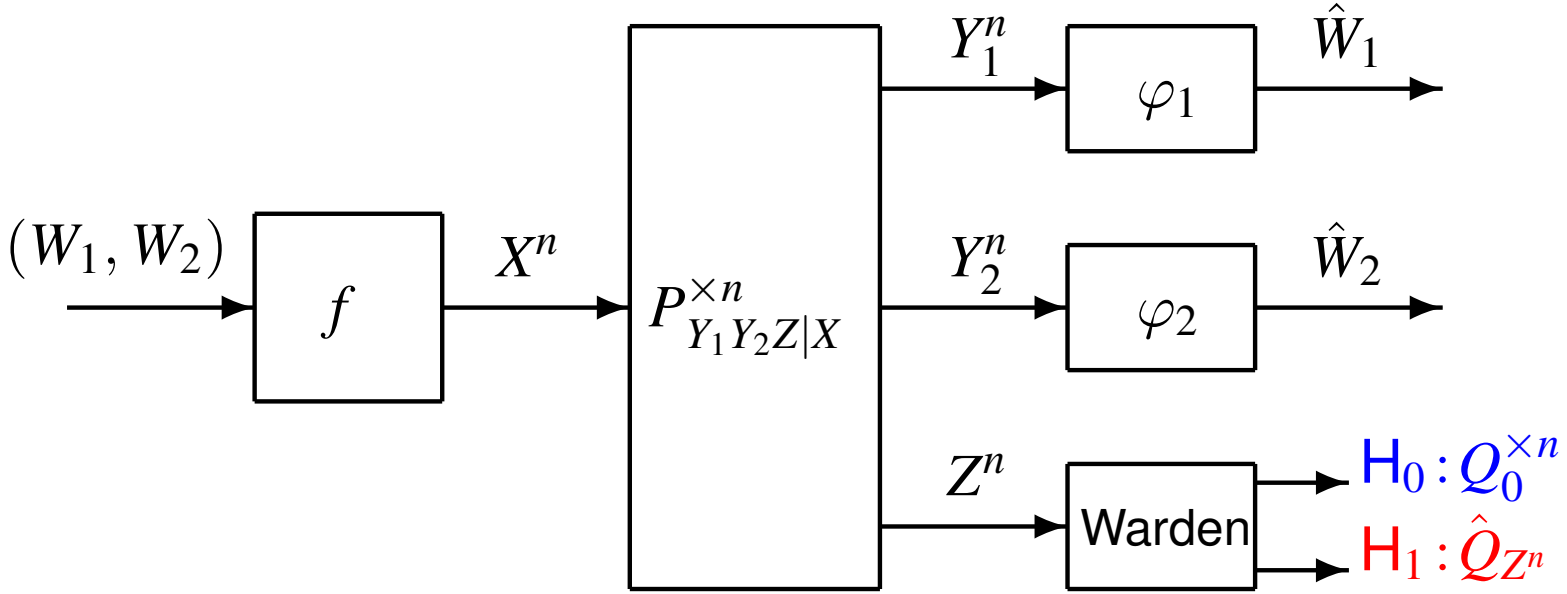

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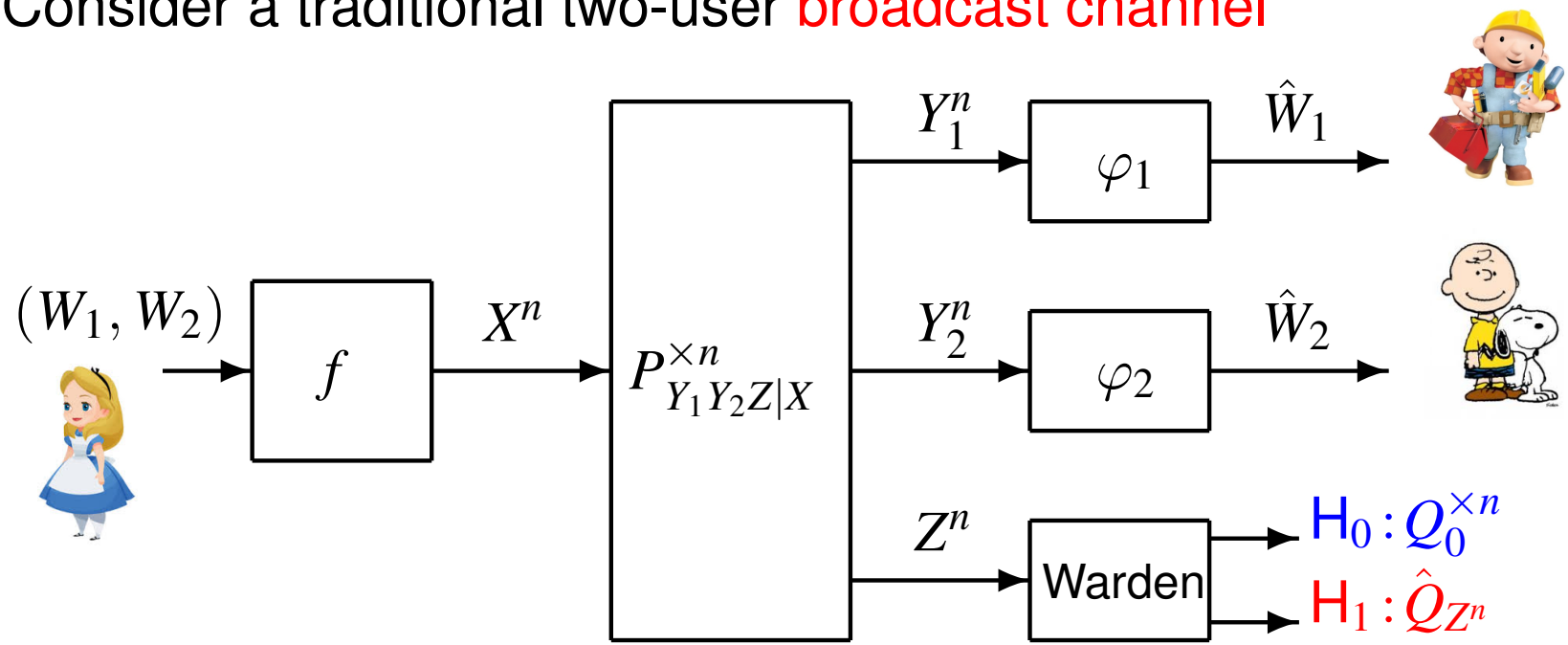

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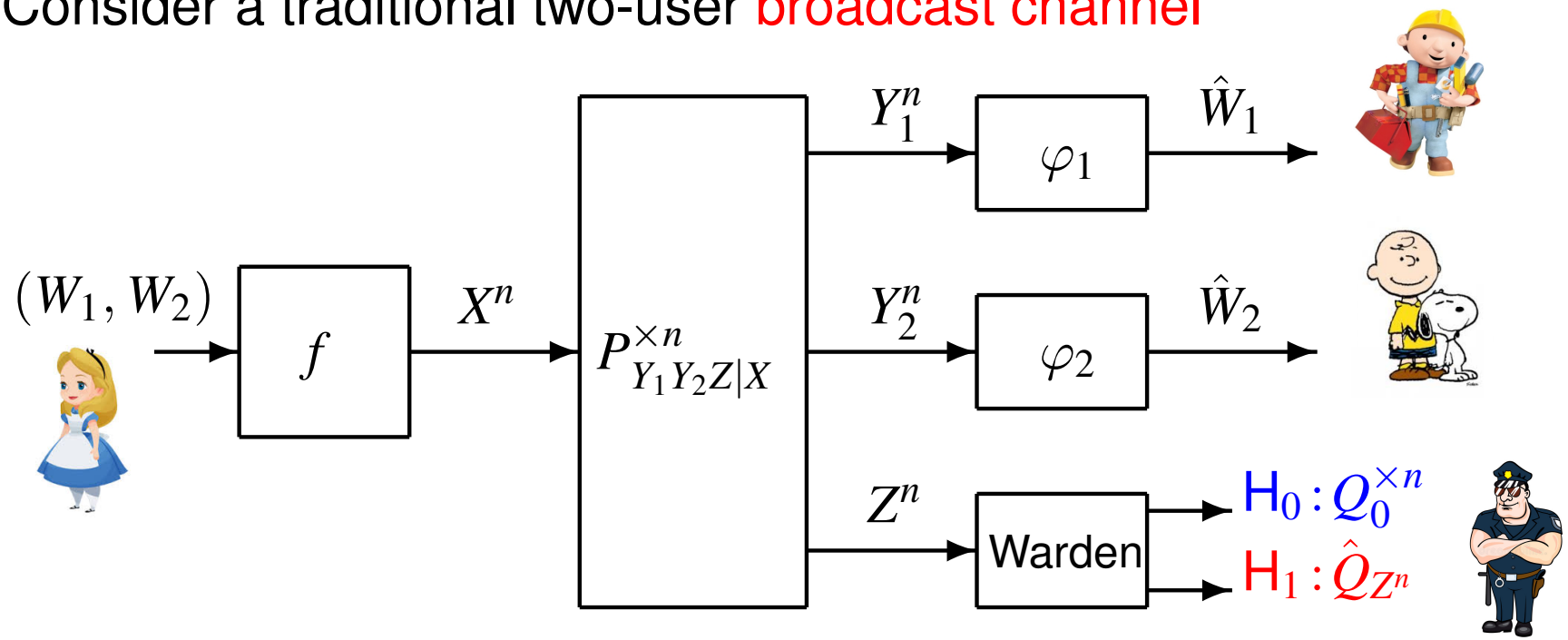

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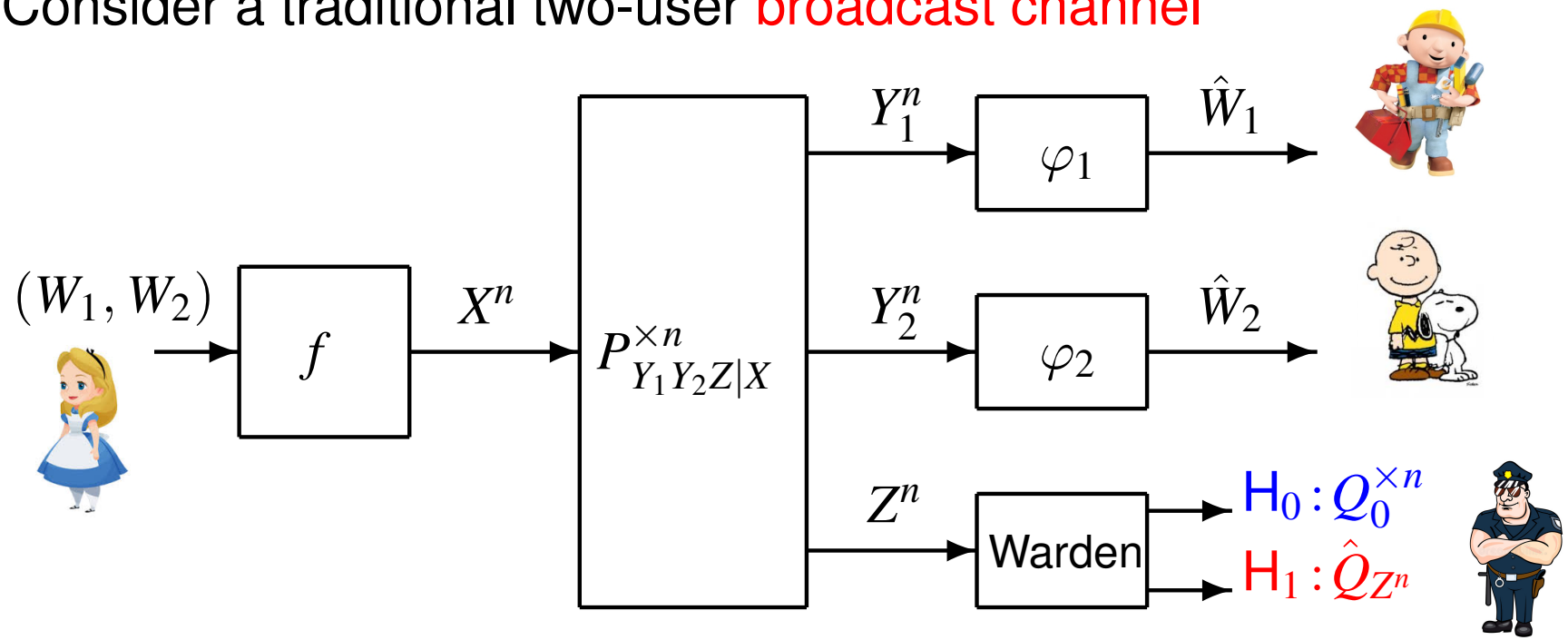

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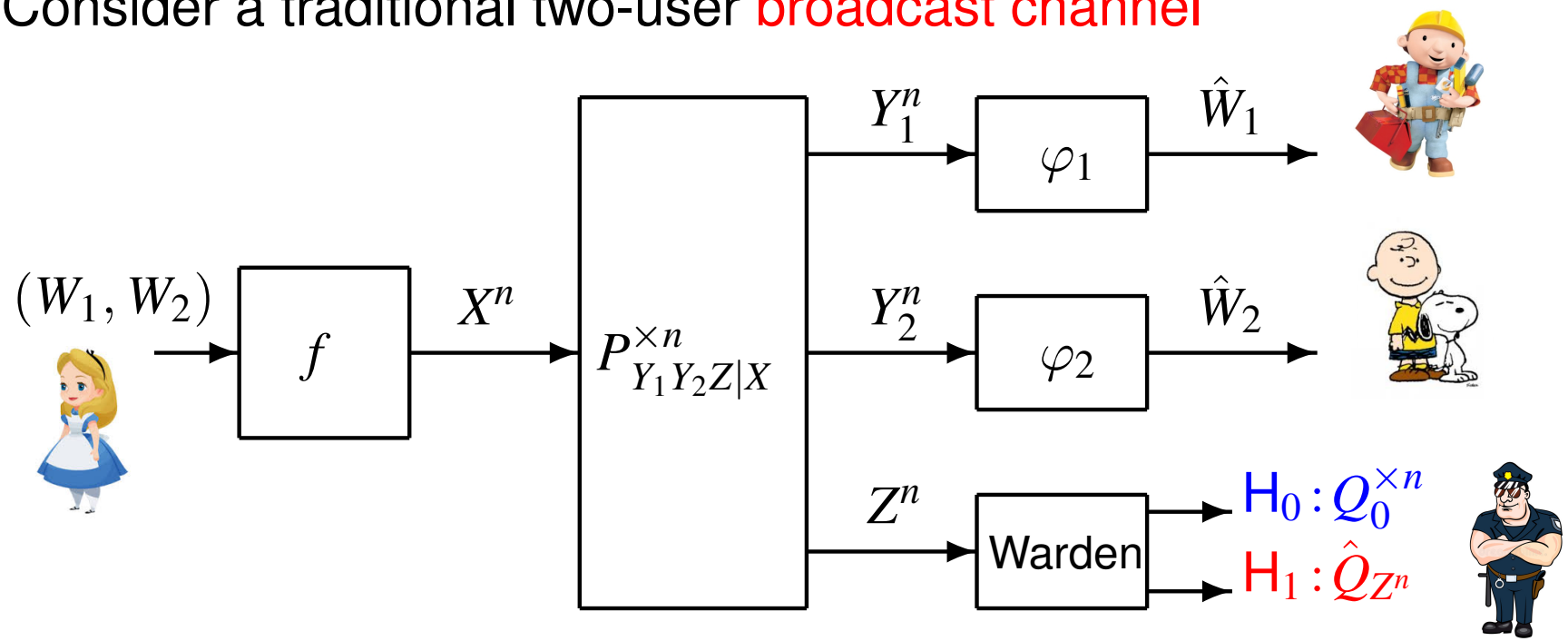

- Covert Communication $\Longleftrightarrow$ Warden should not be able to know whether there is communication
- Should not be able to distinguish between $\hat{Q}_{Z^n}$ (output dist. induced by a code) and the "no-communication" output dist. $Q_0^{\times n}$.

More on Covert Communications over Noisy Channels

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- Optimal performance

$$\pi_{1|0} + \pi_{0|1} = 1 - \frac{1}{2}\left\|Q_0^{\times n} - \hat{Q}_{Z^n}\right\|_1 \geq 1 - \sqrt{D(\hat{Q}_{Z^n}\|Q_0^{\times n})}$$

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- For covert communication, we want to make $D(\hat{Q}_{Z^n}\|Q_0^{\times n})$ small.

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- Renewed interest in fundamental limits of covert communications:
  - Secure space-time codes [Hero '03]
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- Several extensions and related results:
  - Constants in $O(\sqrt{n})$ term [Wang, Wornell, Zheng '16 and Bloch '16]
  - Second-order [Tahmasbi-Bloch '16]
  - Error exponents [Tahmasbi-Bloch-Tan '17]
  - Multi-user [Arumugam-Bloch '16, '17]

Definition of a code

An $(n, M_{1n}, M_{2n}, \varepsilon, \delta)$-code for the broadcast channel with a warden $P_{Y_1,Y_2,Z|X}$ consists of

- Two message sets $\mathcal{M}_j := \{1, \ldots, M_{jn}\}$ for $j = 1, 2$;
- Two independent messages uniformly distributed over their respective message sets, i.e., $W_j \sim \mathrm{Unif}(\mathcal{M}_j)$ for $j = 1, 2$;

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- One encoder $f : \mathcal{M}_1 \times \mathcal{M}_2 \to \mathcal{X}^n$;
- Two decoders $\varphi_j : \mathcal{Y}_j^n \to \mathcal{M}_j$ for $j = 1, 2$;

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such that the following constraints hold:

$$\text{Reliability:} \qquad \Pr\left(\cup_{j=1}^{2} \{\hat{W}_j \neq W_j\} \right) \leq \varepsilon$$

and

$$\text{Covertness:} \qquad D(\hat{Q}_{Z^n} \| Q_0^{\times n}) \leq \delta.$$

Definition of Covert Capacity Region

- $(L_1, L_2) \in \mathbb{R}_+^2$ is (ε, δ)-achievable if there exists a sequence of $(n, M_{1n}, M_{2n}, \varepsilon_n, \delta)$-codes such that

$$\liminf_{n\to\infty} \frac{1}{\sqrt{n\delta}} \log M_{jn} \geq L_j, \quad j = 1, 2,$$

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- The δ-covert capacity region

$$\mathcal{L}_\delta := \bigcap_{\varepsilon\in(0,1)} \mathcal{L}_{\varepsilon,\delta} = \lim_{\varepsilon\to 0} \mathcal{L}_{\varepsilon,\delta}.$$

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- Given a DMC with a warden $P_{Y,Z|X}$, the covert capacity [Wang, Wornell, Zheng '16 and Bloch '16] is

$$L^*(P_{Y,Z|X}) := \sqrt{\frac{2D(W(\cdot|1)\|W(\cdot|0))^2}{\chi_2(Q_1\|Q_0)}}$$

where

$$W(\cdot|x) = P_{Y|X}(\cdot|x) \qquad Q_x := P_{Z|X}(\cdot|x), \quad \forall\, x \in \mathcal{X},$$

and

$$\chi_2(Q_1\|Q_0) := \sum_z \frac{(Q_1(z) - Q_0(z))^2}{Q_0(z)}.$$

Assumption on the BC $P_{Y_1,Y_2,Z|X}$

Assumption

Condition 1: Fix a BC with a warden $P_{Y_1,Y_2,Z|X}$.

Let the covert capacities of $P_{Y_1,Z|X}$ and $P_{Y_2,Z|X}$ be L_1^* and L_2^* respectively.

If $L_1^* \geq L_2^*$ assume that

$$\max_{P_X} \frac{I(X;Y_1)}{I(X;Y_2)} \leq \frac{L_1^*}{L_2^*}.$$

Otherwise, if $L_1^* \leq L_2^*$ assume that

$$\max_{P_X} \frac{I(X;Y_2)}{I(X;Y_1)} \leq \frac{L_2^*}{L_1^*}.$$

Discussion of Condition 1

- Easy to check for binary-input BCs:

$$\frac{D(W_1\|W_0)}{D(V_1\|V_0)} \leq \min_{\gamma\in[0,1]} \frac{D(W_\gamma\|W_0)}{D(V_\gamma\|V_0)}, \qquad W_\gamma(y) = \sum_{x\in\mathcal{X}} P_\gamma(x)W(y|x).$$

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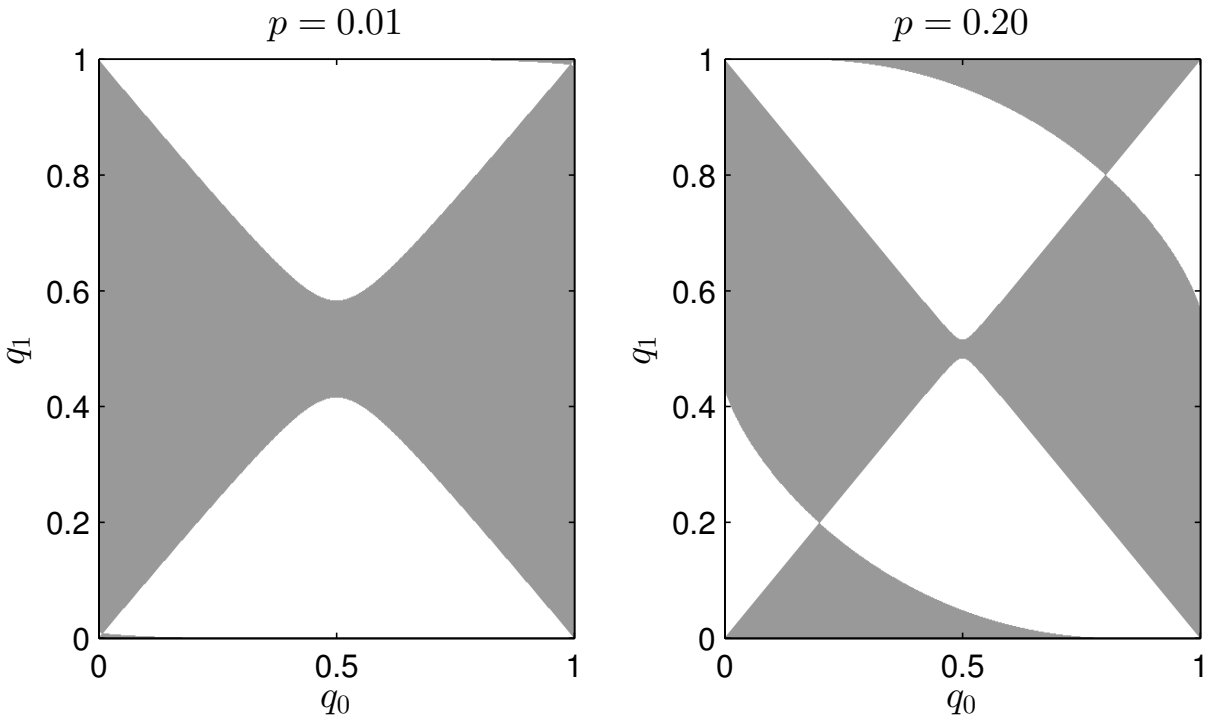
- Let $W = P_{Y_1|X} = \mathrm{BSC}(p)$ and $V = P_{Y_2|X} = \begin{bmatrix} 1-q_0 & q_0 \\ q_1 & 1-q_1 \end{bmatrix}$



Shaded area $\Longrightarrow$ Condition 1 $\surd$

Main Result

Theorem (Tan-Lee (2018))

Assume Condition 1 holds for $P_{Y_1,Y_2,Z|X}$. *For any* $\delta > 0$ *and* $P_{Y_j|X}(\cdot|1) \ll P_{Y_j|X}(\cdot|0)$ *for* $j = 1, 2$,

$$\mathcal{L}_\delta = \left\{ (L_1, L_2) \in \mathbb{R}_+^2 : \frac{L_1}{L_1^*} + \frac{L_2}{L_2^*} \leq 1 \right\}.$$

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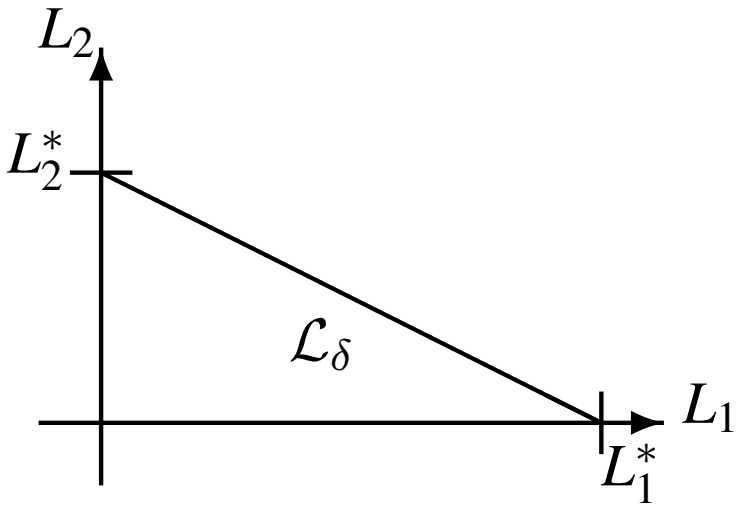

Effect of Key Size

Theorem (Tan-Lee (2018))

Under Condition 1, the tuple $(L_1, L_2, L_{\text{key}})$ *is achievable if and only if*

$$\frac{L_1}{L_1^*} + \frac{L_2}{L_2^*} \leq 1$$

and

$$L_{\text{key}} \geq \left(\frac{L_1}{L_1^*} + \frac{L_2}{L_2^*} \right) L_Z^* - L_1 - L_2,$$

where

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If we operate on the boundary of the keyless covert capacity region,

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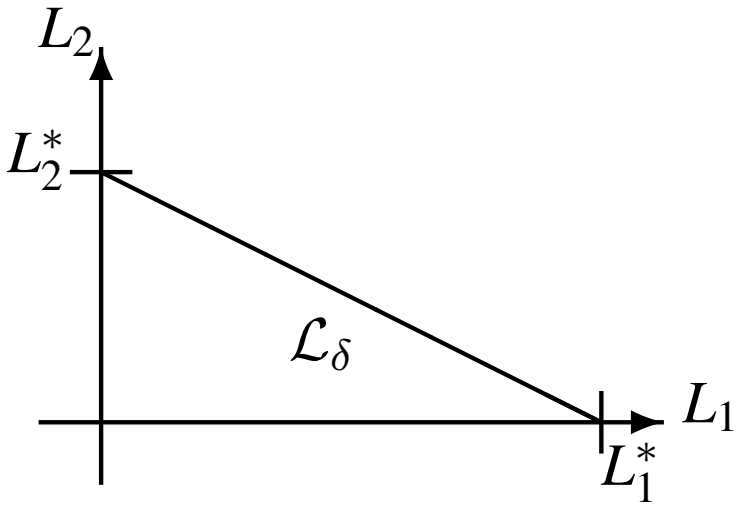
Time-Division is Optimal over Some BCs



- Use an optimal code for $X \to Y_1$ for ρn channel uses. If $\delta' < \delta$,

$$\log M_{1n} \cong \sqrt{\rho n \delta'} L_1^*$$

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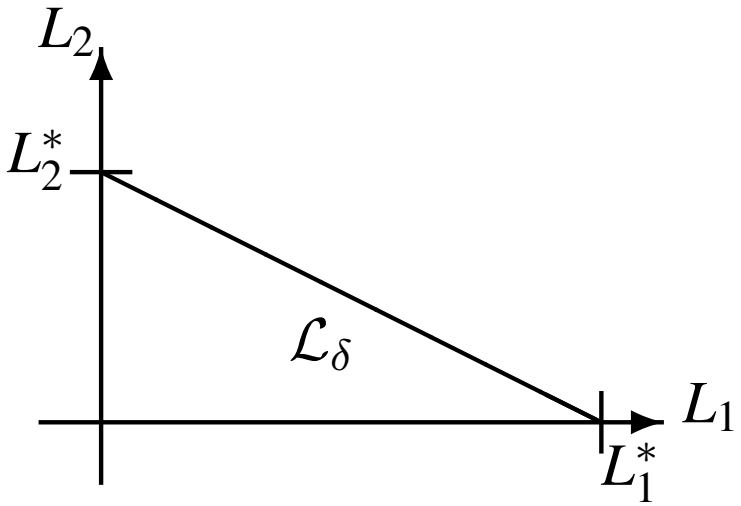

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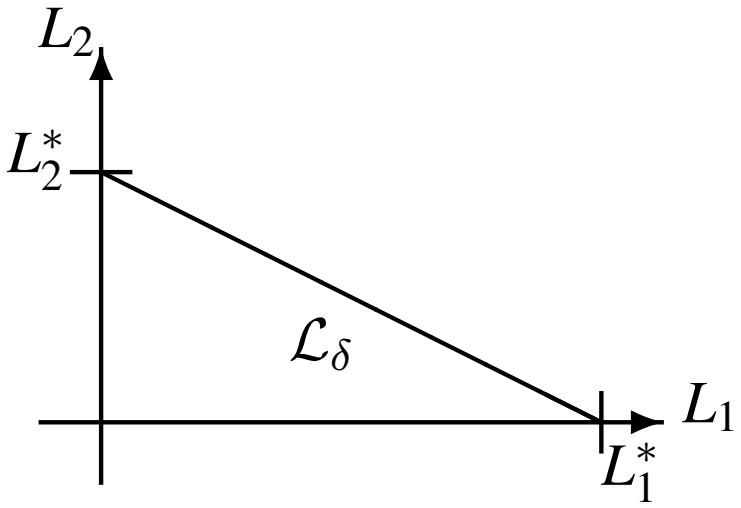

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- Choose $\delta' = (1-\rho)\delta$ achieves the point $(\rho L_1^*, (1-\rho)L_2^*)$.

Time-Division is Optimal for Some BCs: Why?

- Covert communication implies that X^n must have low weight of order $\Theta(\frac{1}{\sqrt{n}})$ [Wang, Wornell, Zheng '16 and Bloch '16], i.e.,

$$|\{i \in [n] : X_i = 1\}| = \Theta(\sqrt{n})$$

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- Same intuition for Gaussian broadcast channels
- But use Entropy Power Inequality instead of Mrs. Gerber's Lemma.

Time-Division is Optimal for Some BCs: Why?

- Superposition coding: Cloud center $u_2^n(w_2)$ carries message w_2; Satellite codeword $x^n(w_1, w_2) = u_1^n(w_1) \oplus u_2^n(w_2)$ carries (w_1, w_2)

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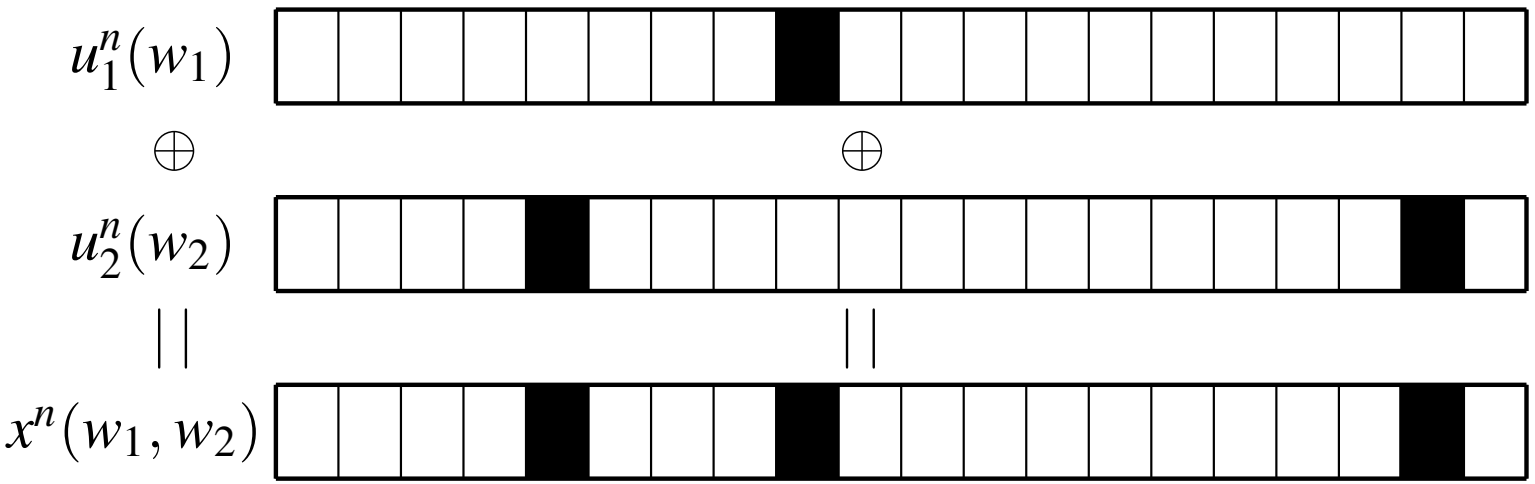

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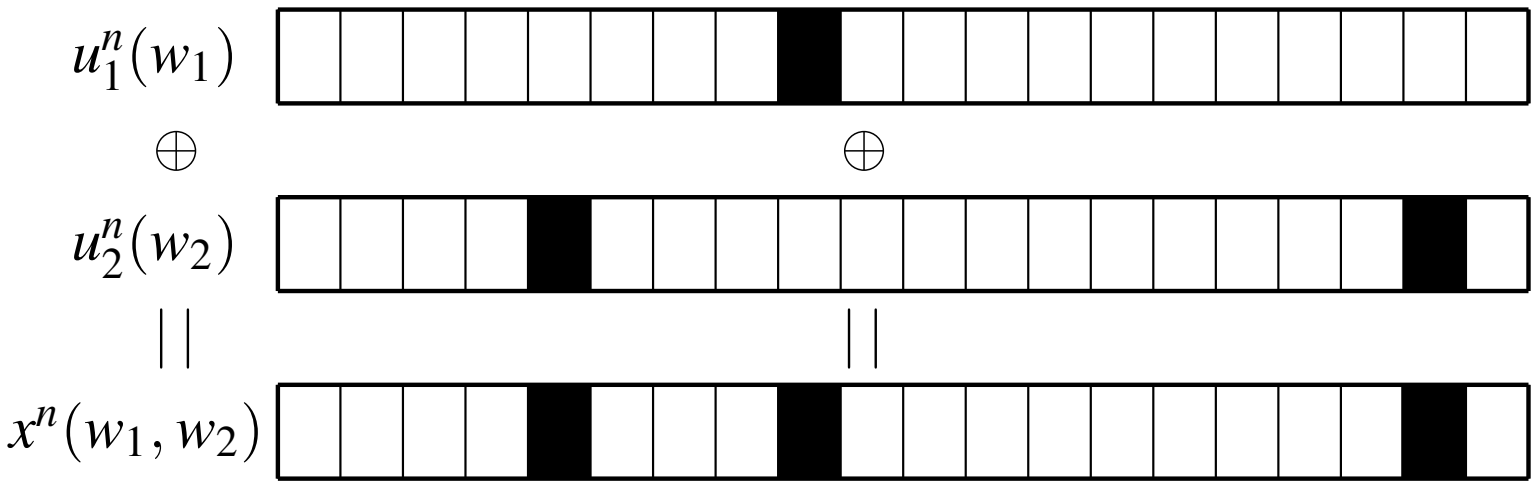

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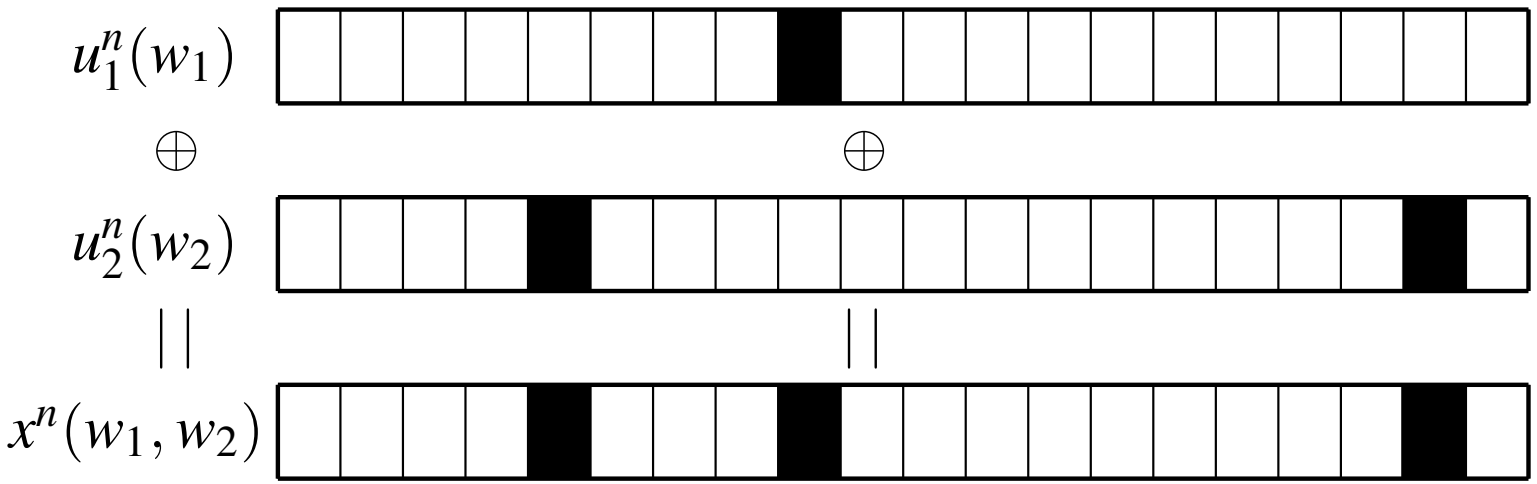

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- Assume weight of $u_1^n(w_1)$ is $\rho\alpha_n$ and that of $u_2^n(w_2)$ is $(1-\rho)\alpha_n$

Time-Division is Optimal for Some BCs: Why?

- Consider BSBCs

$$Y_1 = X \oplus N_1, \qquad Y_2 = X \oplus N_2, \quad N_j \sim \text{Bern}(p_j), \quad p_2 \geq p_1$$

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- Put

$$\text{wt}(u_1^n(w_1)) = \rho\alpha_n n, \quad \text{and} \quad \text{wt}(u_2^n(w_2)) = (1-\rho)\alpha_n n,$$

the superposition coding inner bound with $X = U_1 \oplus U_2$ reads $(U_2 - X - Y_1 - Y_2)$

$$R_1 \leq I(X; Y_1|U_2) = I(U_1; Y_1 \oplus N_1) \approx \rho\alpha_n L_1^*$$
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- Hence, we can write

$$\frac{R_1}{L_1^*} + \frac{R_2}{L_2^*} \lessapprox \alpha_n$$

El Gamal's Converse for More Capable BCs

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where U_{1i} and U_{2i} satisfy $(U_{1i}, U_{2i}) - X_i - (Y_{1i}, Y_{2i})$.

Converse Proof: Upper Bound on λ-Sum Rate

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- For the BS-BC, Mrs. Gerber's Lemma [Wyner-Ziv (1973)] helps to simplify

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- Now,

$$\max_{P_{U|X}} [I(X;Y_1|U) - \lambda I(X;Y_2|U)] = \mathbb{C}[I(P_X,W) - \lambda I(P_X,V)]$$

where $W = P_{Y_1|X}$ and $V = P_{Y_2|X}$ and the concave envelope is defined as

$$\mathbb{C}[f](x) := \inf\{g(x) : g \geq f, g \text{ is concave}\}$$

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- The usual superposition coding region is

$$\mathcal{C} = \bigcup_{P_X, P_{U|X}} \left\{ (R_1, R_2) \in \mathbb{R}_+^2 \mid R_1 \leq I(X;Y_1|U), R_2 \leq I(U;Y_2) \right\}$$

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$$\mathcal{C} = \bigcap_{\lambda \geq 1} \left\{ (R_1, R_2) \in \mathbb{R}_+^2 \mid R_1 + \lambda R_2 \leq \max_{P_X} \lambda I(X;Y_2) + T_\lambda(X) \right\}$$

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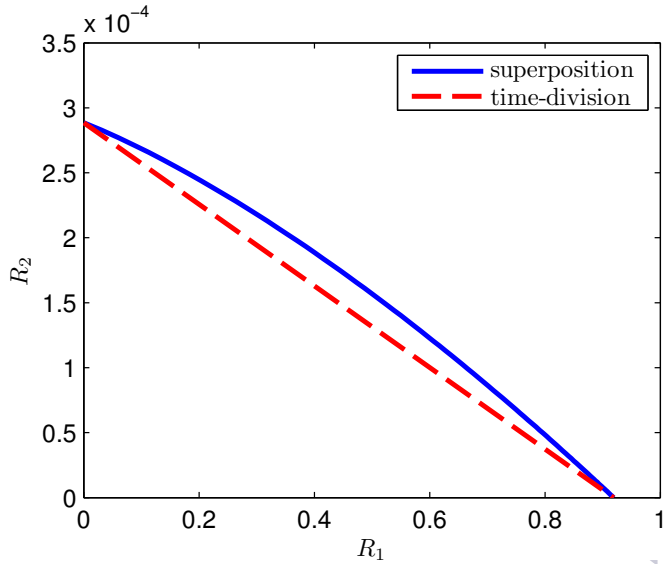

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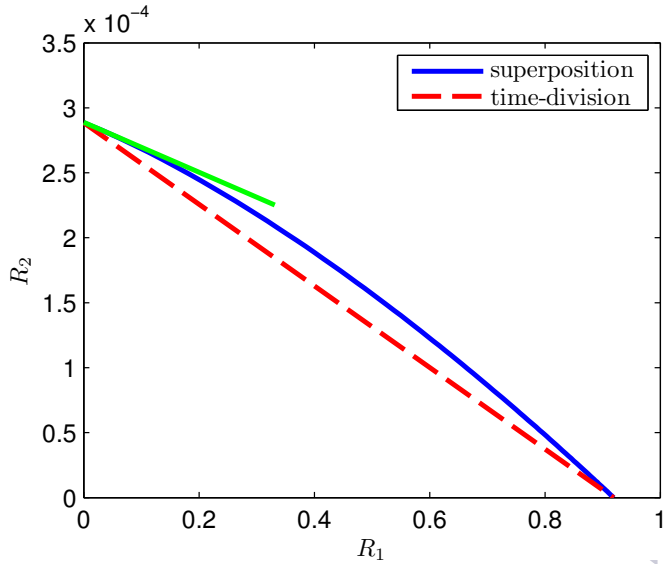

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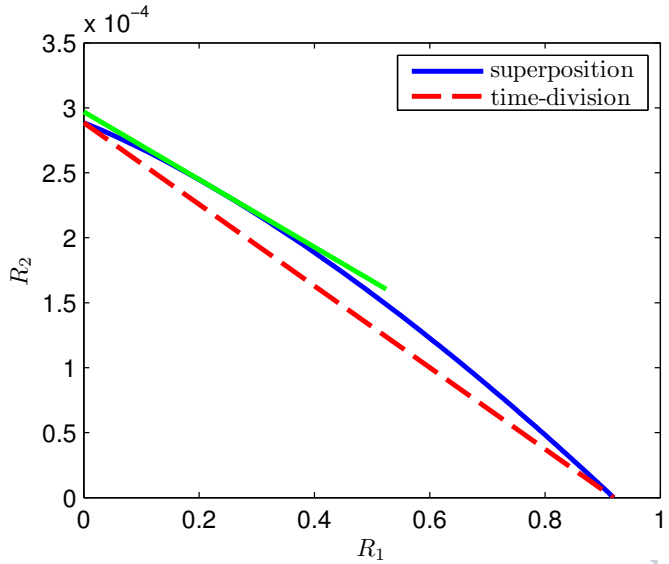

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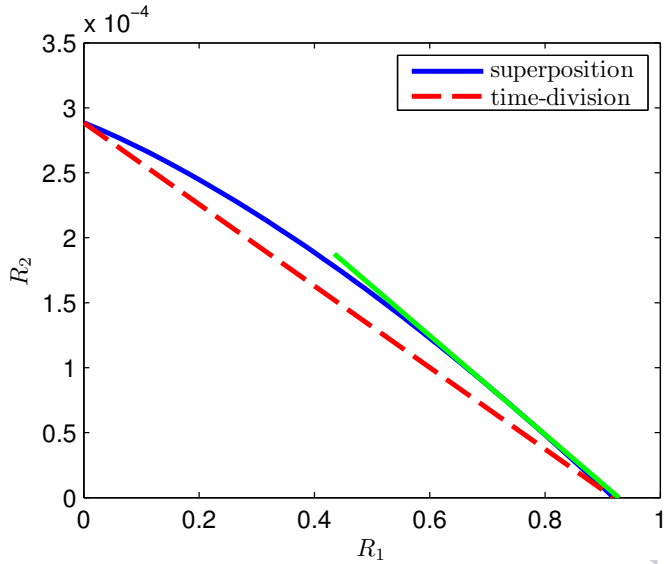

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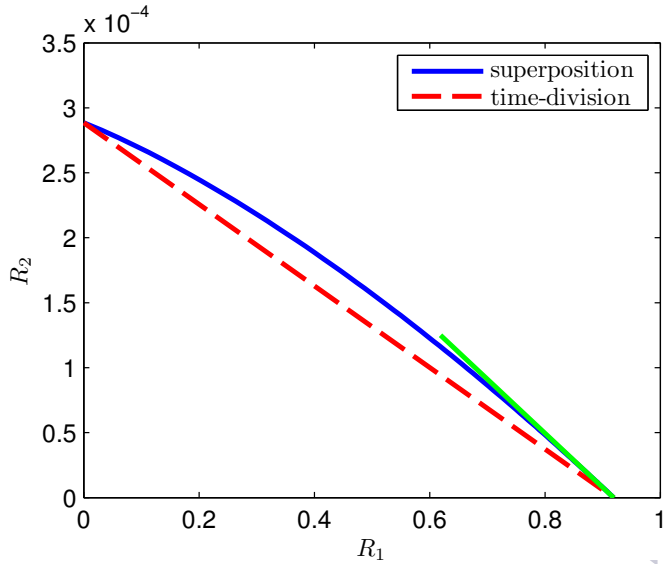

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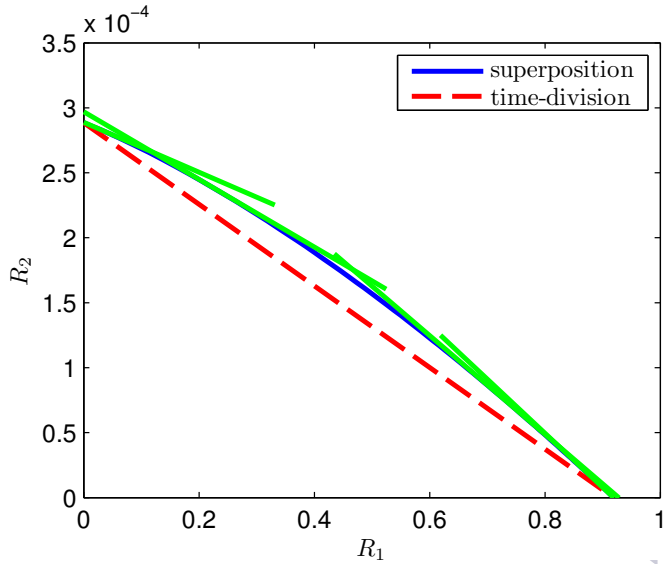

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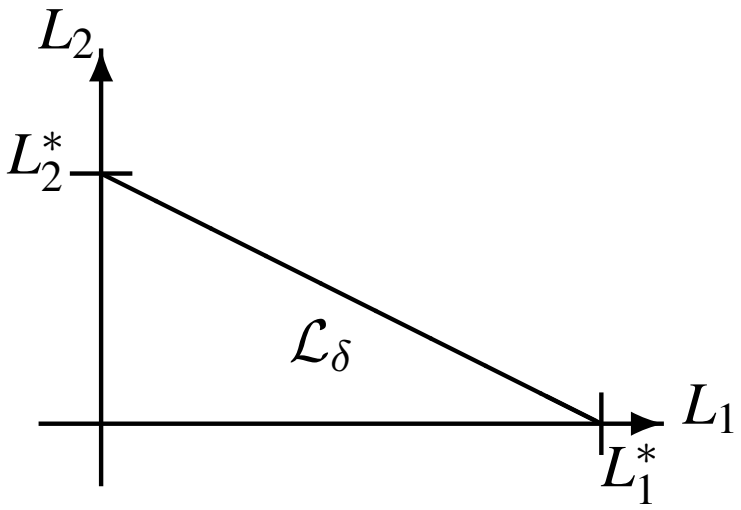
- Finally, recalling that $\lambda = L_1^*/L_2^*$,

$$L_1 + \frac{L_1^*}{L_2^*} \cdot L_2 \lessapprox \frac{L_1^*}{L_2^*} \cdot L_2^* = L_1^*, \quad \Longrightarrow \quad \frac{L_1}{L_1^*} + \frac{L_2}{L_2^*} \leq 1.$$

Conclusion and Open Problems



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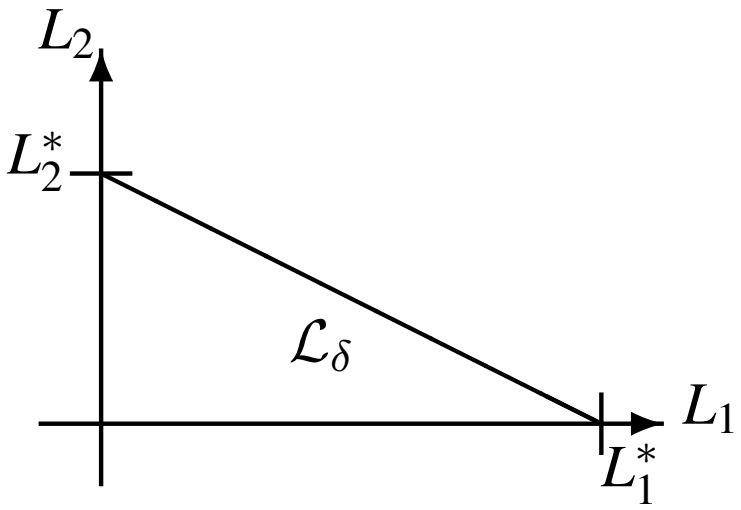

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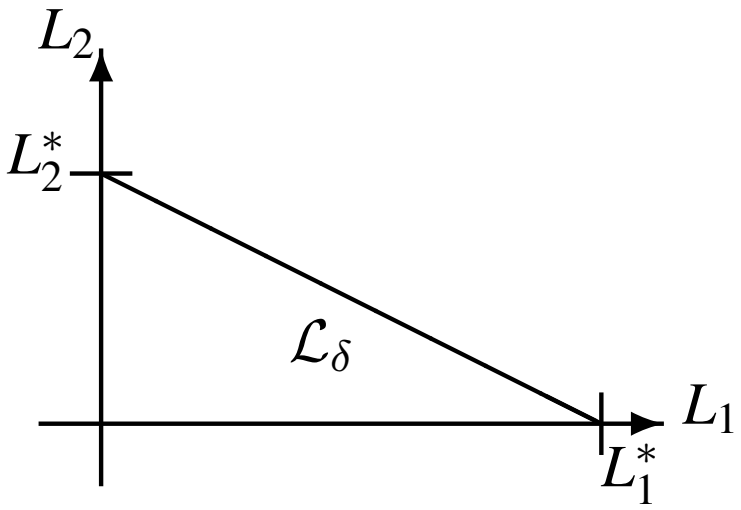

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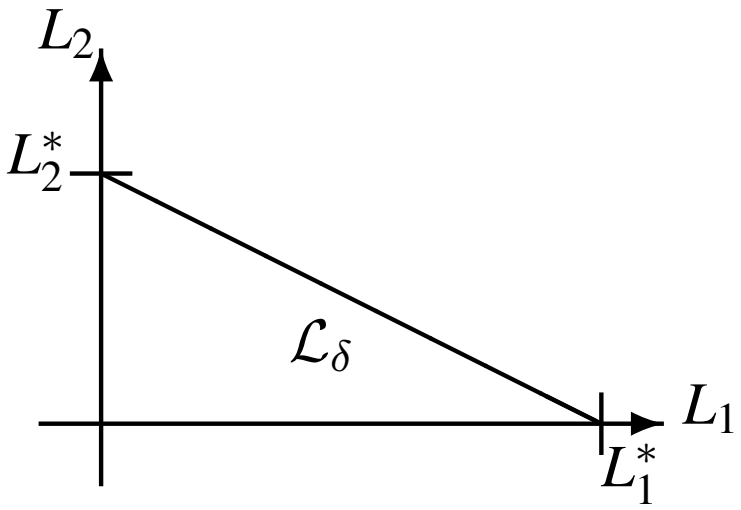

- Concave envelope representation of bounds on capacity region with auxiliary RVs is very useful
- What can we say about BCs which don't satisfy Condition 1?
- More than 2 legitimate receivers?

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The Department of Electrical and Computer Engineering (ECE) at the National University of Singapore (NUS) is offering positions for postdoctoral fellows who will work closely with Dr. Vincent Tan at the intersection of information theory, statistical signal processing, and machine learning. Some sample topics include:

- Fundamental performance limits (and algorithms) for dictionary learning (e.g., matrix factorization), ranking, and deep learning architectures;
- Learning in the presence of privacy constraints;
- Learning in the large alphabet regime;
- Learning of graphical models and other statistical models.

Working in traditional topics in Shannon's information theory of interest to the PI will also be highly encouraged. Some sample topics include:

- Multi-user information theory;
- Strong converse and second-order asymptotics;
- Error exponent analysis and the method of types;
- Information-theoretic security;