Sample Complexity for Topology Estimation in Networks of LTI Systems

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CDC (Dec 2011)

Motivation I

- Tree network $\mathcal{T} = (\mathcal{V}, \mathcal{E})$ with p nodes and $p - 1$ directed arcs

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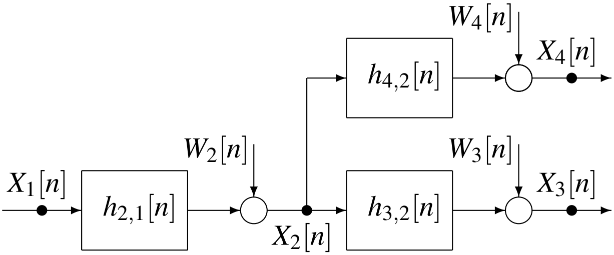

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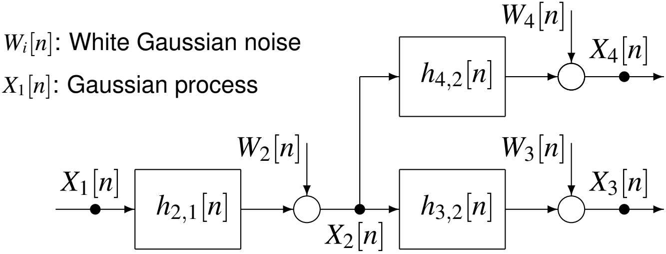

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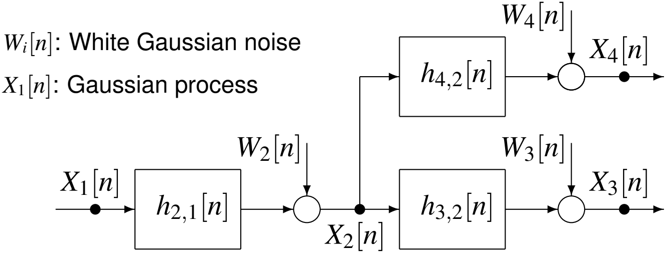

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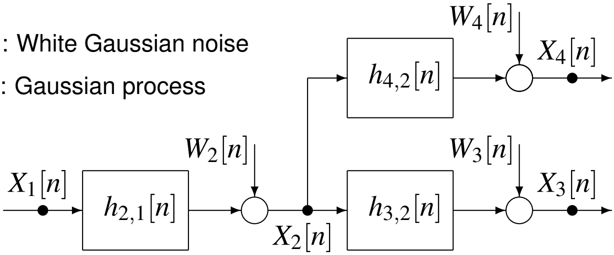
$W_i[n]$: White Gaussian noise

$X_1[n]$: Gaussian process



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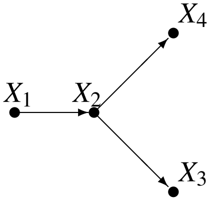

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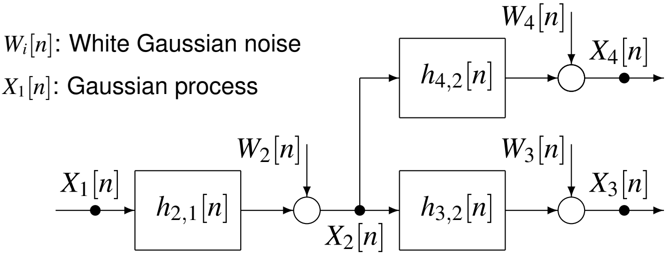

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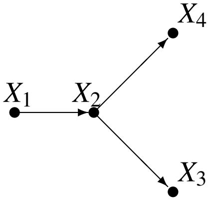
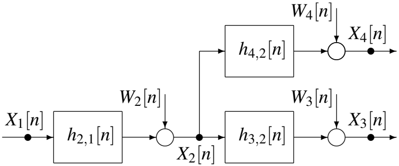


$$P_{X_1,X_2,X_3,X_4} = P_{X_1} P_{X_2|X_1} P_{X_3|X_2} P_{X_4|X_2}$$

Motivation II



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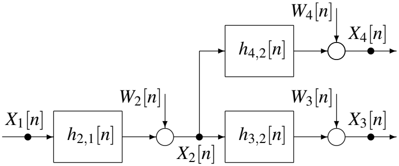

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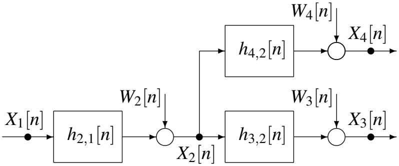

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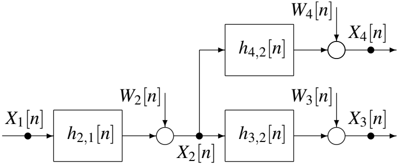

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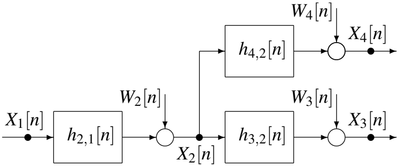

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- Many applications in system identification and model selection

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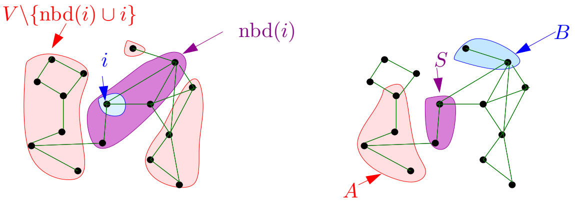

$X_i \perp\!\!\!\perp \mathbf{X}_{V\backslash\{\mathrm{nbd}(i)\cup i\}} | \mathbf{X}_{\mathrm{nbd}(i)}$ $\mathbf{X}_A \perp\!\!\!\perp \mathbf{X}_B | \mathbf{X}_S$

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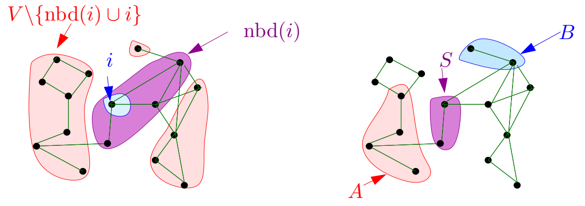

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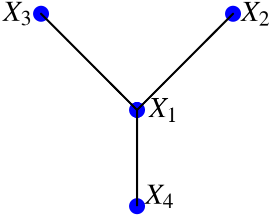
- Kalman filter, hidden Markov models, Bayesian networks...

Tree-Structured Graphical Models



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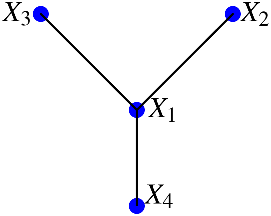

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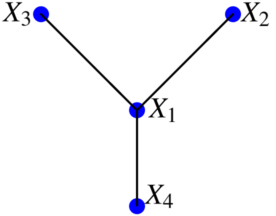

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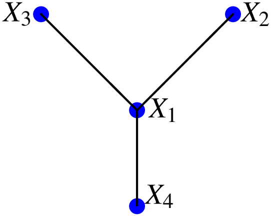

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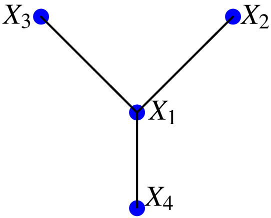

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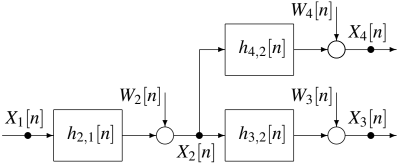
- Inference on Trees is tractable
  - Sum-Product Algorithm (Pearl 1988)

Main Result



Let us observe the p Gaussian WSS processes for N time steps

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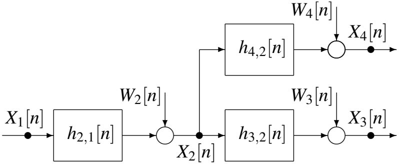

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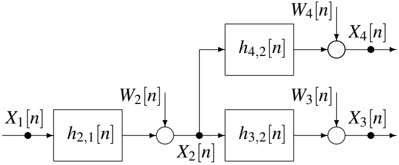

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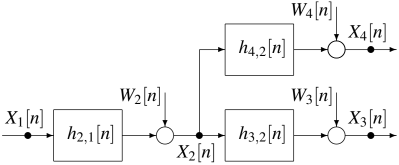

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$$N = O\left(\log^{1+\varepsilon}\left(\frac{p}{\delta^{1/3}}\right)\right), \qquad \textit{then} \qquad \mathbb{P}(\hat{\mathcal{T}}_N \neq \mathcal{T}) < \delta$$

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- Given observations $\{X_1[n], \ldots, X_p[n]\}_{n=0}^{N-1}$, learn the tree $\mathcal{T}$

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$$\min_{\nu \in \mathcal{M}_{\text{Tree}}} D(\mu \,||\, \nu), \qquad D(\mu \,||\, \nu) := \int \log \frac{d\mu}{d\mu} \, d\nu$$

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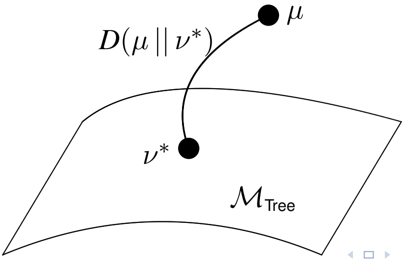

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Lemma (Chow-Liu for processes)

The measure that achieves the minimum is Markov on $\hat{\mathcal{T}}_N$ given by

$$\hat{\mathcal{T}}_N = \operatorname*{arg\,max}_{\mathcal{T} \in \textit{Tree}} \sum_{(i,j) \in \mathcal{T}} I_\mu(X_i; X_j),$$

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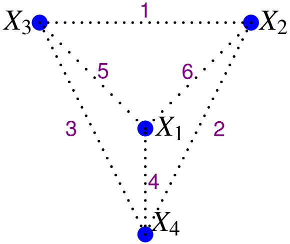
- Max-weight spanning tree problem: Kruskal's algorithm
- The mutual information rate for Gaussian processes is

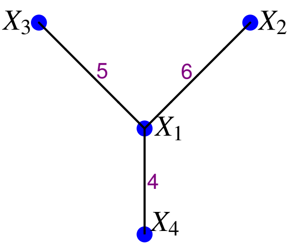
$$I(X_i; X_j) := -\frac{1}{4\pi} \int_0^{2\pi} \log\left(1 - |\gamma_{i,j}(\omega)|^2\right) \, d\omega$$

- $\gamma_{i,j}(\omega)$ is the coherence function between X_i and X_j

Learning Procedure III

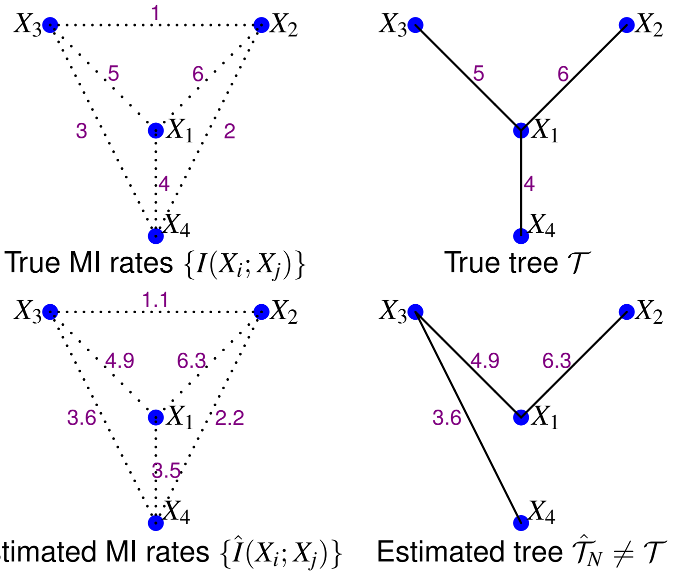


True MI rates $\{I(X_i; X_j)\}$



True tree $\mathcal{T}$

Learning Procedure III



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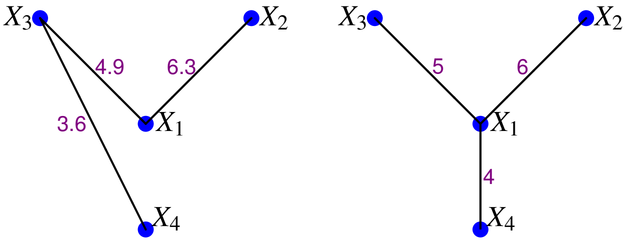

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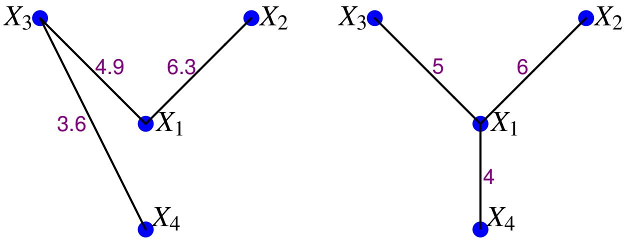

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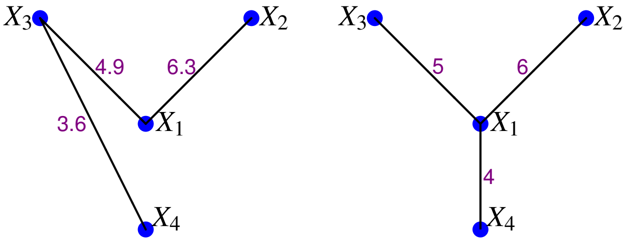

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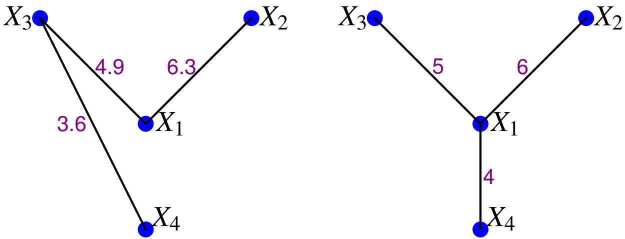

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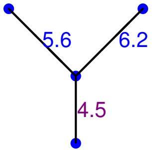
- How do errors occur?
- The order of the estimated MI rates relative to the true ones is important

Order of MI Rates

Correct Structure

$I(X_i; X_j)$	6	5	4	3	2	1
$\hat{I}(X_i; X_j)$	6.2	5.6	4.5	2.8	2.2	1.1



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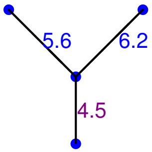
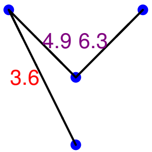


Incorrect Structure!

$I(X_i;X_j)$	6	5	4	3	2	1
$\hat{I}(X_i;X_j)$	6.3	4.9	3.5	3.6	2.2	1.1



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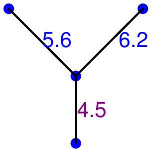

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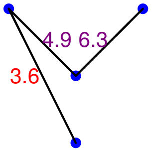

Structure Unaffected

$I(X_i;X_j)$	6	5	4	3	2	1
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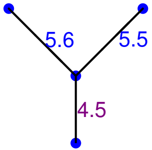

Order of MI Rates

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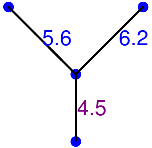

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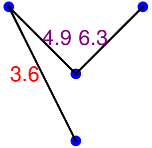

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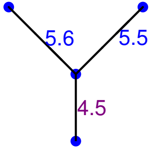

Estimate MI rates and ensure $\hat{I}(X_i;X_j) \approx I(X_i;X_j)$

Estimating the Mutual Information Rates I

- Given $\{X_1[n]\}_{n=0}^{N-1}, \ldots, \{X_p[n]\}_{n=0}^{N-1}$, estimate $\{I(X_i; X_j)\}_{(i,j) \in \mathcal{V}}$ using Bartlett's procedure

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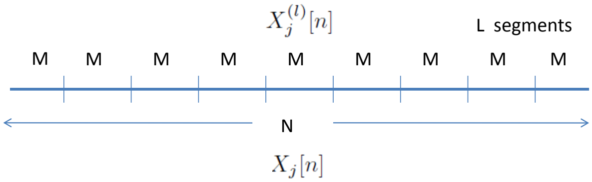
- Divide each length-N realization $\{X_j[n]\}_{n=0}^{N-1}$ into L non-overlapping segments of length M such that $LM \approx N$



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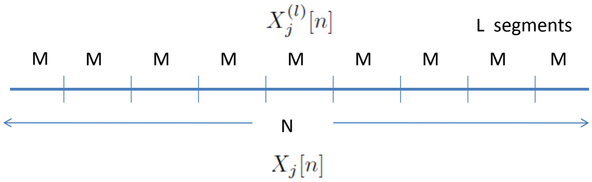

- Compute the length-M discrete Fourier transform for each signal segment, i.e., $\tilde{X}_j^{(l)}[k] = \mathcal{F}(X_j^{(l)}[n])$

Estimating the Mutual Information Rates II

- Estimate the time-averaged periodograms using Bartlett's averaging procedure on the L signal segments, i.e.,

$$\hat{\Phi}_{X_i}[k] := \frac{1}{L}\sum_{l=0}^{L-1}\left|\tilde{X}_i^{(l)}[k]\right|^2, \qquad \hat{\Phi}_{X_i,X_j}[k] := \frac{1}{L}\sum_{l=0}^{L-1}\left(\tilde{X}_i^{(l)}[k]\right)^* \tilde{X}_j^{(l)}[k]$$

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- Estimate the MI rates by using the Riemann sum:

$$\hat{I}(X_i;X_j) := -\frac{1}{2M}\sum_{k=0}^{M-1}\log\left(1-|\hat{\gamma}_{i,j}[k]|^2\right).$$

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- For convenience, we let $M = M_L$ be a function of L

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We can choose $M = \lceil L^{\varepsilon} \rceil, \varepsilon > 0$ to satisfy above condition

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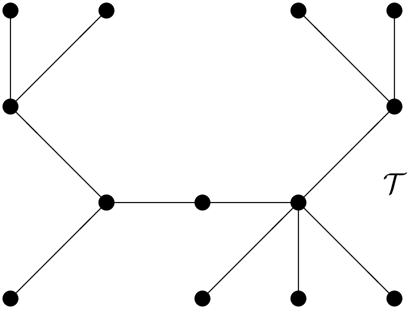
  where $\hat{I}(X_i; X_j)$ are the estimated mutual information rates

- How to get $\mathbb{P}(\hat{\mathcal{T}}_N \neq \mathcal{T})$ from the constituent error probabilities $\mathbb{P}\left(|\hat{I} - I| > \eta\right)$?

Bounding the Overall Error Probability II



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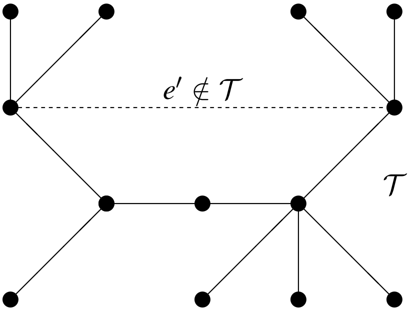

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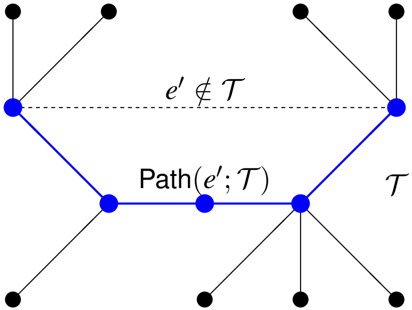

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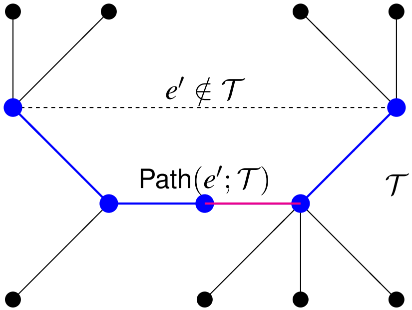

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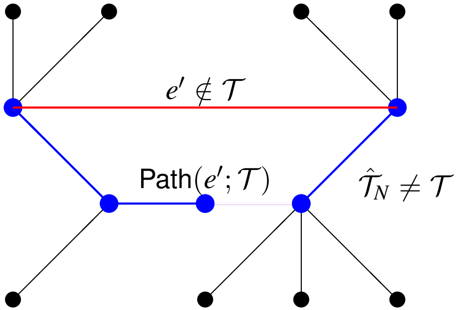

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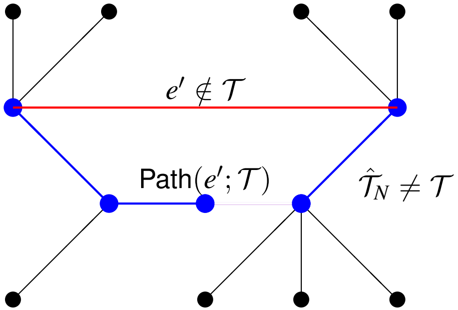

Lemma

$$\left\{\hat{\mathcal{T}}_N \neq \mathcal{T}\right\} = \bigcup_{(k,l)\notin\mathcal{E}} \; \bigcup_{(i,j)\in \mathit{Path}(k,l)} \left\{\hat{I}(X_k; X_l) \geq \hat{I}(X_i; X_j)\right\}$$

Possible to bound $\mathbb{P}(\hat{\mathcal{T}}_N \neq \mathcal{T})$ by using the union bound

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