Information-Theoretic Limits for Streaming Communication

Vincent Y. F. Tan

Joint work with Si-Hyeon Lee and Ashish Khisti (Toronto)

National University of Singapore (NUS)

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Outline

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1. Background and Streaming Setup

2. Achievability Results and Proof Sketches

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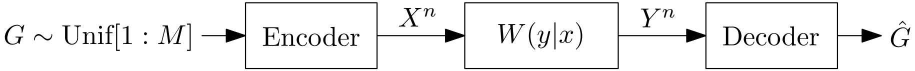
Block-Coding Formalism



Block channel coding

- Consider a discrete memoryless channel $W : \mathcal{X} \to \mathcal{Y}$

Block-Coding Formalism

$$G \sim \mathrm{Unif}[1:M] \longrightarrow \boxed{\text{Encoder}} \xrightarrow{X^n} \boxed{W(y|x)} \xrightarrow{Y^n} \boxed{\text{Decoder}} \longrightarrow \hat{G}$$

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Block channel coding

- Consider a discrete memoryless channel $W : \mathcal{X} \to \mathcal{Y}$
- Fundamental problem in information theory is the interplay between n, R and ε
- Shannon (1948) showed that the maximum rate of communication with $\varepsilon \to 0$ as $n \to \infty$ is

$$C(W) = \max_{P_X} I(X;Y) = \max_{P_X} I(P_X, W)$$

Refined Asymptotics

Regime	Large deviations [Gallager '65] [Shannon et al. '67]	Moderate deviations [Altüg, Wagner '14] [Polyanskiy, Verdú '10]	Central limit [Strassen '62] [Hayashi '09] [Polyanskiy et al. '10]
Code Rate	$R < C$	$R = C - \rho_n$ $\rho_n \to 0$ and $n\rho_n^2 \to \infty$	$R \approx C - \sqrt{\frac{V}{n}} Q^{-1}(\varepsilon)$
Error Prob.	Exponential $\varepsilon \le \exp\{-nE_r(R)\}$	Subexponential $\varepsilon \approx \exp\left\{-\frac{n\rho_n^2}{2V}\right\}$	Non-vanishing $\varepsilon > 0$

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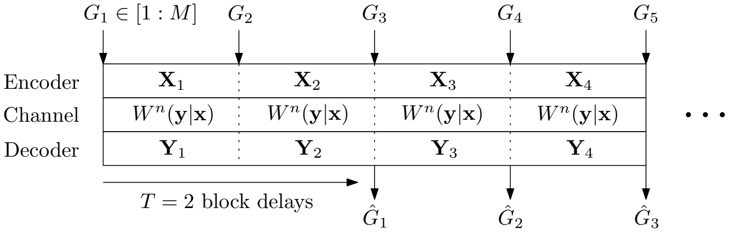
- Channel dispersion

$$V = V(W) = \min_{P_X : I(P_X, W) = C(W)} \mathrm{var}(i(X;Y))$$

- Smaller dispersion is better.

Streaming Services

An Information-Theoretic Model



Streaming setup

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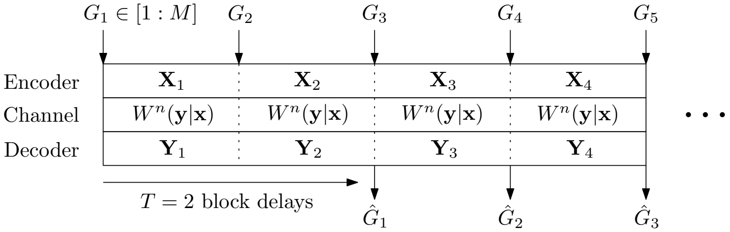


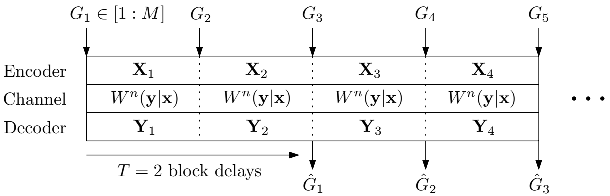
Streaming setup

- An (n, M, ε, T)-streaming code consists of
  1. a sequence of messages $\{G_k\}_{k \ge 1}$ each uniform over $\mathcal{G} = [1:M]$;
  2. a sequence of encoders $\phi_k : \mathcal{G}^k \to \mathcal{X}^n$ s.t. $\phi_k(G^k) = \mathbf{X}_k$;
  3. a sequence of decoders $\psi_k : \mathcal{Y}^{(k+T-1)n} \to \mathcal{G}$ s.t. $\psi_k(\mathbf{Y}^{k+T-1}) = \hat{G}_k$.

that satisfies

$$\limsup_{N \to \infty} \sum_{k=1}^{N} \frac{\Pr(\hat{G}_k \neq G_k)}{N} \le \varepsilon.$$

Streaming Setup



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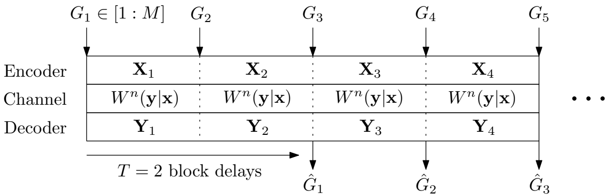

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  Use codeword only for fresh msg vs. also for previous msges?

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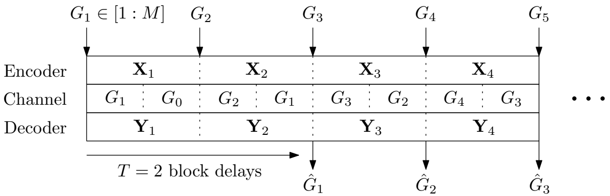

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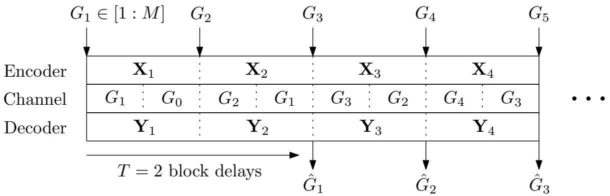

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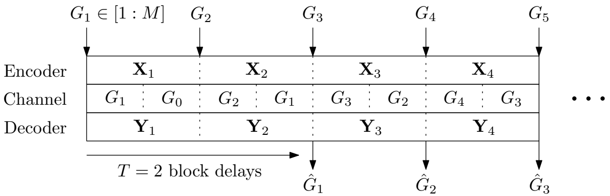

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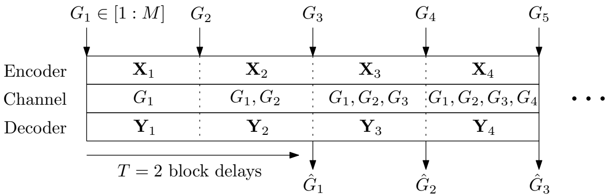

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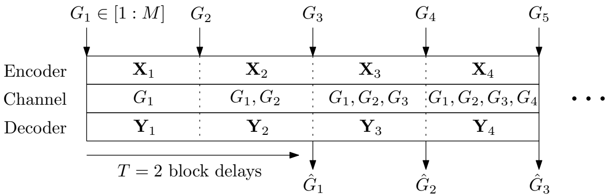

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- Joint encoding?
  - Does improve

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First Main Result (Achievability)

Theorem (Lee-T.-Khisti (2015))

Let the message size grow as

$$\log M_n = n(C - \rho_n)$$

where

$$\rho_n \geq 0, \quad \rho_n \to 0, \quad n\rho_n^2 \to \infty.$$

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There exists a sequence of $(n, M_n, \varepsilon_n, T)$*-streaming codes such that*

$$\varlimsup_{n\to\infty} \frac{1}{n\rho_n^2} \log \varepsilon_n \leq -\frac{T}{2V}$$

Interpretation of Moderate Deviations Result

$$\varlimsup_{n\to\infty} \frac{1}{n\rho_n^2} \log \varepsilon_n \le -\frac{T}{2V}$$

- In block coding,

$$\lim_{n\to\infty} \frac{1}{n\rho_n^2} \log \varepsilon_n^* = -\frac{1}{2V}$$

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- Dispersion V is reduced by a factor of T

Second Main Result (Achievability)

Theorem (Lee-T.-Khisti (2015))

For any $L > 0$, let the message size grow as

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Second Main Result (Achievability)

Theorem (Lee-T.-Khisti (2015))

For any $L > 0$, let the message size grow as

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Then there exists a sequence of $(n, M_n, \varepsilon_n, T)$-streaming codes s.t.

$$\varepsilon_n \lesssim c \cdot Q\left(\sqrt{\frac{T}{V}}L\right) \qquad c \approx 1.$$

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- Dispersion V is approx. reduced by a factor of T

Summary of Main Results

Regime	Moderate deviations	Central limit
Operating rate	$R = C - \rho_n$ $\rho_n \to 0$ and $n\rho_n^2 \to \infty$	$R = C - \frac{L}{\sqrt{n}}$ $L > 0$
Error Prob.	$\varepsilon \approx \exp\left\{-\frac{Tn\rho_n^2}{2V}\right\}$	$\varepsilon \approx Q\left(L\sqrt{\frac{T}{V}}\right)$
	$V \to V/T$	
Encoding	Joint encoding of previous and fresh msges	
Decoding	Sequential decoding of previous and new msges	
	Accumulation of error probabilities	
Key innovation	Non-asymptotic moderate deviations theorem	Truncated memory structure

Recap of Coding Scheme for Block Coding

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- Decoding: If there exists a unique $g \in [1:M]$ such that

$$i(\mathbf{x}(g);\mathbf{y}) > \log M, \quad \text{where} \quad i(\mathbf{x};\mathbf{y}) = \log \frac{W^n(\mathbf{y}|\mathbf{x})}{(P_X W)^n(\mathbf{y})}$$

  let $\hat{G} = g$.

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  let $\hat{G} = g$.
- Error analysis:

$$\varepsilon \le \Pr(\mathcal{E}_1) + \Pr(\mathcal{E}_2)$$

  where

$$\mathcal{E}_1 := \{i(\mathbf{X}(G);\mathbf{Y}) \le \log M\}$$
$$\mathcal{E}_2 := \{\exists\, \tilde{g} \ne G \text{ s.t. } i(\mathbf{X}(\tilde{g});\mathbf{Y}) > \log M\}$$

  Note that $\mathcal{E}_1$ is dominant in both regimes

Analysis of Error Probability

- Probability of error

$$\varepsilon_n \approx \Pr\left(i(\mathbf{X}(G);\mathbf{Y}) \le M\right) = \Pr\left(\sum_{l=1}^{n} Z_l \le \log M\right)$$

where

$$Z_l := \log \frac{W(Y_l|X_l)}{P_X W(Y_l)}, \quad l = 1, \ldots, n$$

are i.i.d. random variables.

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- Under various regimes, analyze

$$\Pr\left(\sum_{l=1}^{n} Z_l \le \log M\right).$$

Analysis of Error: Moderate Deviations Regime

Theorem (Moderate Deviations Theorem (Dembo and Zeitouni))

Under regularity conditions on Z_l, *and* $\log M = n(C - \rho_n)$,

$$\lim_{n\to\infty} \frac{1}{n\rho_n^2} \log \Pr\left(\sum_{l=1}^{n} Z_l \le \log M\right) = -\frac{1}{2V}.$$

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- However, note that the standard MD theorem is asymptotic in nature
- We need a non-asymptotic version in the streaming scenario

Analysis of Error: Central Limit Regime

Theorem (Berry-Esseen Theorem)

Under regularity conditions on Z_l, and

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we have

$$\Pr\left(\sum_{l=1}^{n} Z_l \le \log M\right) = Q\left(\frac{L}{\sqrt{V}}\right) \pm \frac{\tau}{\sqrt{n}}.$$

where τ is a constant (depending on Z_1).

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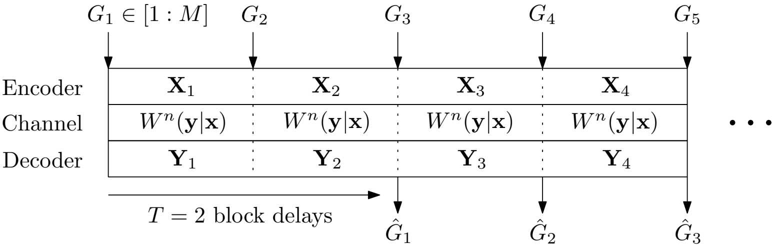
- Thus, we have
$$\varepsilon_n \le Q\left(\frac{L}{\sqrt{V}}\right) + O\left(\frac{1}{\sqrt{n}}\right).$$
- However, note that the Berry-Esseen residual terms hurt us in the streaming setup

[Streaming setup] Moderate deviations regime



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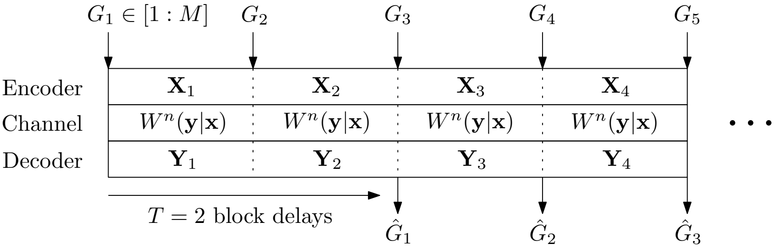

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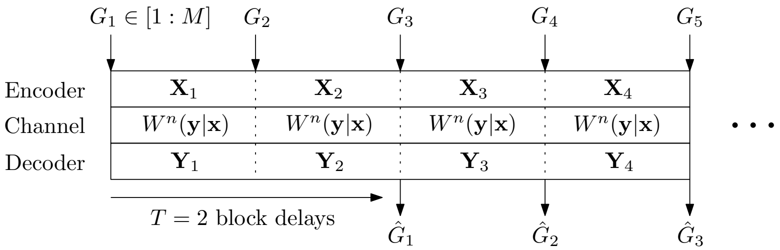

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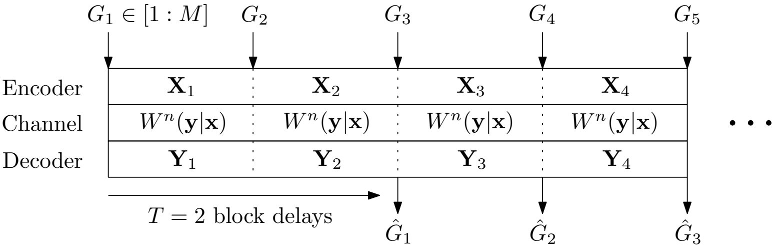

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- Encoding at block k: Send $\mathbf{x}_k(G_1, \cdots, G_k)$.
- Decoding at block $k+1$:
  - Target message: G_k
  - Due to joint encoding, G_k is in error if any of $\hat{G}_1, \cdots, \hat{G}_{k-1}$ is in error.
  - Sequentially decode $G_1, \cdots, G_k$.

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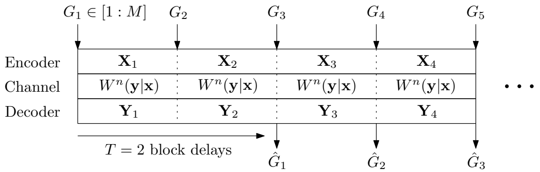

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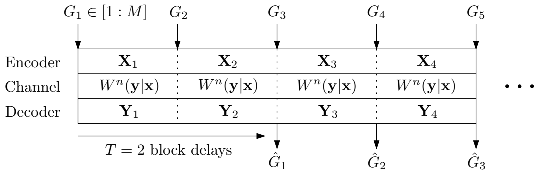

- $T = 2$: At the end block 3, sequentially decode G_1 and G_2.
- Re-decode G_1: Choose $\hat{G}_1'$ as a unique $g_1 \in [1:M]$ such that

$$i([\mathbf{x}_1(g_1)\ \mathbf{x}_2(g_1, g_2)\ \mathbf{x}_3(g_1, g_2, g_3)], [\mathbf{y}_1\ \mathbf{y}_2\ \mathbf{y}_3]) > 3\log M, \quad \text{for some} \quad g_2, g_3$$

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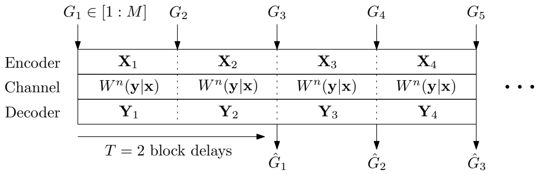

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- Decode G_2: Choose $\hat{G}_2$ as a unique $g_2 \in [1:M]$ such that

$$i([\mathbf{x}_2(\hat{G}_1', g_2)\ \mathbf{x}_3(\hat{G}_1', g_2, g_3)], [\mathbf{y}_2\ \mathbf{y}_3]) > 2\log M \quad \text{for some} \quad g_3$$

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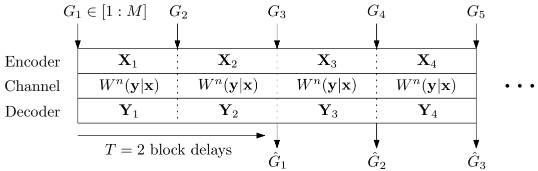

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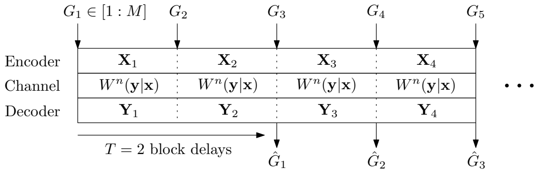
[Streaming Setup] Moderate Deviations Regime



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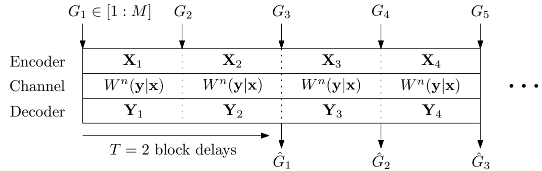

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- Cannot "exchange limits"

- Need to develop a non-asymptotic upper bound for moderate deviations theorem [Altüg and Wagner (2014)]

[Streaming Analysis] Moderate Deviations Regime

Lemma

Under regularity conditions on Z_l, for any positive ρ_n satisfying $\rho_n \to 0$ and $n\rho_n^2 \to \infty$,

$$\Pr\left(\frac{1}{n}\sum_{l=1}^{n} Z_l \geq \rho_n\right) \leq \exp\left\{-n\left(\frac{\rho_n^2}{2\sigma^2} - \frac{\rho_n^3}{6\sigma^6}K\right)\right\}$$

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Using the lemma, we conclude that for all $k \in \mathbb{N}$,

$$\varlimsup_{n\to\infty} \frac{1}{n\rho_n^2} \log\left[\varlimsup_{n\to\infty} \frac{1}{N}\sum_{k=1}^{N} \Pr(\hat{G}_k \neq G_k)\right] \leq -\frac{T}{2V}.$$

[Streaming Analysis] Central limit regime

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- Compared to the block coding case, $n \to jn$, $L \to \sqrt{j}L$.

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&= \sum_{j=T}^{\infty} Q\left(\frac{\sqrt{j}L}{\sqrt{V}}\right) + \sum_{j=T}^{\infty} \frac{\tau}{\sqrt{jn}} \\
&\le c \cdot Q\left(\frac{\sqrt{T}L}{\sqrt{V}}\right) + \sum_{j=T}^{\infty} \frac{\tau}{\sqrt{jn}},
\end{aligned}
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  where $c \approx 1$ for a wide range of channel parameters.

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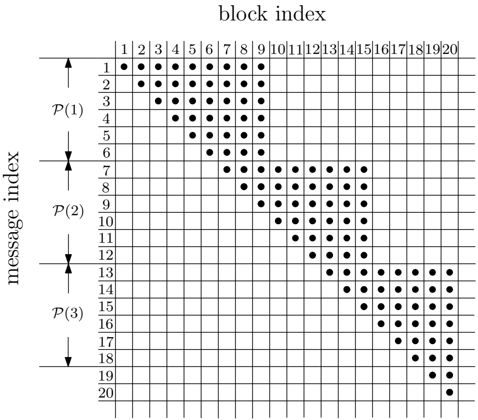
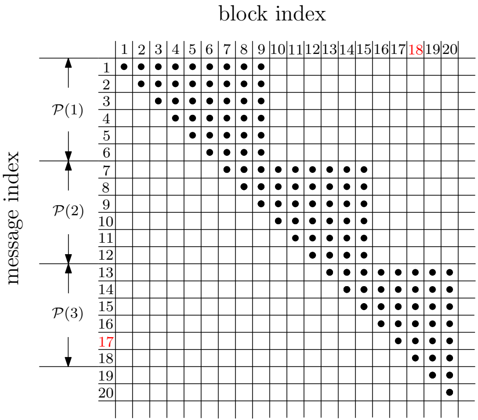
- The remainder terms from the Berry-Esseen theorem diverge!



Memory structure with $A = 9$, $B = 4$

- A, B: Max/Min memories
- Decode all msgs in the previous group and all previous msgs in the current group
- Example of $A = 9$, $B = 4$, $T = 2$ To decode G_{17} at the end of block 18, decodes $G_7, \cdots, G_{17}$ by considering codewords in blocks $10, \cdots, 18$.
- Judiciously choose A and B as functions of n to balance
  - Rate penalty ($\frac{B}{A} \downarrow$)
  - Contributions to error probability
    Remainder terms ($A \downarrow$)
    Previous group ($B \uparrow$)

[Streaming Analysis] Truncated Memory



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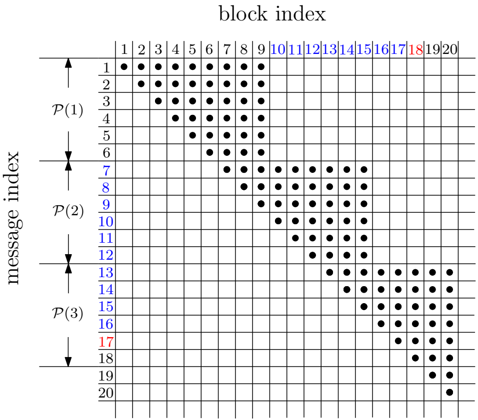

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Extension 1: Erasure Option

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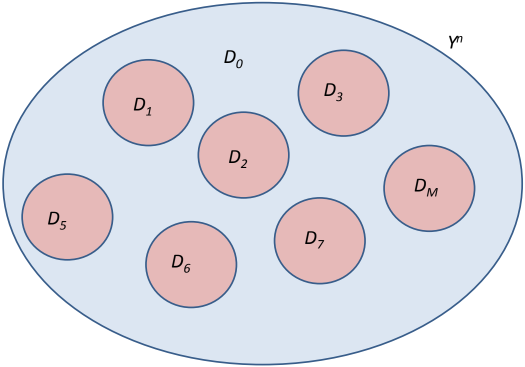
- Seek upper bounds on ε and ε' when M is the moderate deviations regime.

Illustration of the Erasure Option



Decoding with an erasure option

Result for Erasure Option

Theorem

Let the message size grow as

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$$\varlimsup_{n\to\infty} \frac{1}{n\rho_n^2} \log \varepsilon_n \leq -\frac{T(1-\gamma)^2}{2V}$$

$$\varlimsup_{n\to\infty} \frac{1}{n\rho_n} \log \varepsilon_n' \leq -T\gamma$$

for any $0 < \gamma < 1$.

Discussion of Result for Erasure Option

- The undetected error probability is

$$\varepsilon_n \le \exp\left\{-n\rho_n^2 \cdot \frac{T(1-\gamma)^2}{2V} + o(n\rho_n^2)\right\}$$

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Result for Decoding with Variable Delay

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There exists a sequence of $(n, M_n, \varepsilon_n, T_n)$*-streaming codes with average delay constraint such that*

$$\lim_{n\to\infty} T_n = T$$

$$\overline{\lim_{n\to\infty}} \frac{1}{n\rho_n} \log \varepsilon_n \leq -T$$

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- A significant gain in the can be achieved in the moderate deviations regime with streaming and variable delay without feedback

Outline

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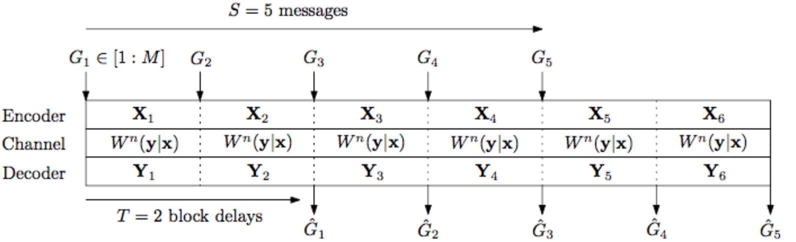
  2. a sequence of encoding functions
  $$\phi_k : \mathcal{G}^{\min\{k,S\}} \to \mathcal{X}^n \quad \text{for} \quad k \in [1:S+T-1]$$
  3. a sequence of decoding functions
  $$\psi_k : \mathcal{Y}^{(k+T-1)n} \to \mathcal{G}, \quad \text{for} \quad k \in [1:S]$$

  s.t. the maximum error probability over all S msgs satisfies
  $$\max_{k \in [1:S]} \Pr(\hat{G}_k \neq G_k) \leq \varepsilon.$$

A Slightly Different Streaming Setup



$T = 2$ and $S = 5$. A total of five msges ($S = 5$) are sequentially encoded and are sequentially decoded after the delay of two blocks ($T = 2$).

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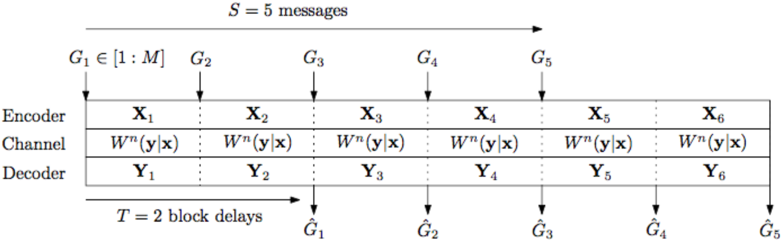

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Fundamental limit on error probability

$$\varepsilon^*(n, M, T, S) := \min\{\varepsilon : \exists \text{ an } (n, M, \varepsilon, T, S)\text{-streaming code}\}$$

Main Converse Result

Theorem (Lee-T.-Khisti (2016))

For an output symmetric DMC with $V > 0$, consider sequences M_n and S_n such that

$$\log M_n = n(C - n^{-t}), \quad \textit{with} \quad 0 < t < 1/3,$$

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Matches previous moderate deviations achievability result

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$$E^+(R;W) = E_{sp}(R;W)$$

for output symmetric channels, where

$$E^+(R;W) := \min_{V:C(V)\le R} \max_P D(V\|W|P) \qquad \text{(Haroutunian)}$$

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- Range $0 < t < 1/3$ is more restrictive than the usual $0 < t < 1/2$

Discussion of Main Converse Result

$$\lim_{n\to\infty} \frac{1}{n^{1-2t}} \log \varepsilon^*(n, M_n, T, S_n) = -\frac{T}{2V}$$

- Need to restrict to output symmetric channels because

$$E^+(R;W) = E_{sp}(R;W)$$

for output symmetric channels, where

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- Range $0 < t < 1/3$ is more restrictive than the usual $0 < t < 1/2$
- Range of $S_n = \omega(n^t) \cap \exp(o(n^{1-2t}))$ is rather extensive

Proof Ideas for the Converse Part

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- Step 1: Assume a feedforward decoder (genie-aided decoder)
- Step 2: Lower bound maximal error probability over a certain number of messages S_n^* using an auxiliary channel V_n^*
- Step 3: Lower bound error probability of the maximal error message under true channel W using a change-of-measure idea due to Sahai (2008)

Step 1 of the Converse Part

Definition

A feedforward decoder consists of a sequence of decoding function $\psi_k^f : \mathcal{G}^{k-1} \times \mathcal{Y}^{(k+T-1)n} \to \mathcal{G}$ for $k \in [1 : S_n]$, i.e.,

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Lemma

For a feedforward decoder, there exists a sequence of decoding functions $\psi_k^* : \mathcal{G}^{k-1} \times \mathcal{Y}^{Tn} \to \mathcal{G}$ for $k \in [1 : S_n]$, i.e.,

$$\psi_k^*(G^{k-1}, \mathbf{Y}_k^{k+T-1}) = \hat{G}_k \quad \textit{and satisfies}$$

$$\Pr(G_k \neq \psi_k^*(G^{k-1}, \mathbf{Y}_k^{k+T-1})) \leq \Pr(G_k \neq \psi_k^f(G^{k-1}, \mathbf{Y}^{k+T-1}))$$

Step 2 of the Converse Part

Lemma

Let V_n^* be an auxiliary channel defined as

$$V_n^* := \min_{V : C(V) \le R_n - \delta_n} \max_P D(V \| W | P)$$

for appropriately chosen $R_n = C - n^{-t}$ and $\delta_n = o(n^{-t})$.

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$$\max_{k \in [1:S_n]} \Pr\left(\hat{G}_k \neq G_k\right) \ge \delta_n'.$$

Thus $\exists$ at least a fraction of $\delta_n'/2$ messages s.t.

$$(V_n^*)^{Tn}\left(\{\textit{bad channel outputs }\} \middle| \textit{ cwd given message}\right) \ge \frac{\delta_n'}{2}$$

Step 3 of the Converse Part

Lemma

If for some $x^{Tn} \in \mathcal{X}^{Tn}$ *with type* $\hat{P}_{x^{Tn}}$*,*

$$(V_n^*)^{Tn}(\mathcal{A}|x^{Tn}) \geq \frac{\delta_n'}{2}, \quad \textit{for some} \quad \mathcal{A} \subset \mathcal{Y}^{Tn},$$

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$$W^{Tn}(\mathcal{A}|x^{Tn}) \geq \frac{\delta_n'}{4} \exp\left\{-Tn\left(D(V_n^* \| W | \hat{P}_{x^{Tn}}) + \eta_n\right)\right\}$$

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Finally approximate

$$\max_P D(V_n^*\|W|P) = E^+(R_n - \delta_n) = E_{sp}(C - n^{-t} - o(n^{-t})) \leq \frac{n^{-2t}}{2V} + o(n^{-2t}).$$

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- Joint encoding and decoding of fresh and previous messages
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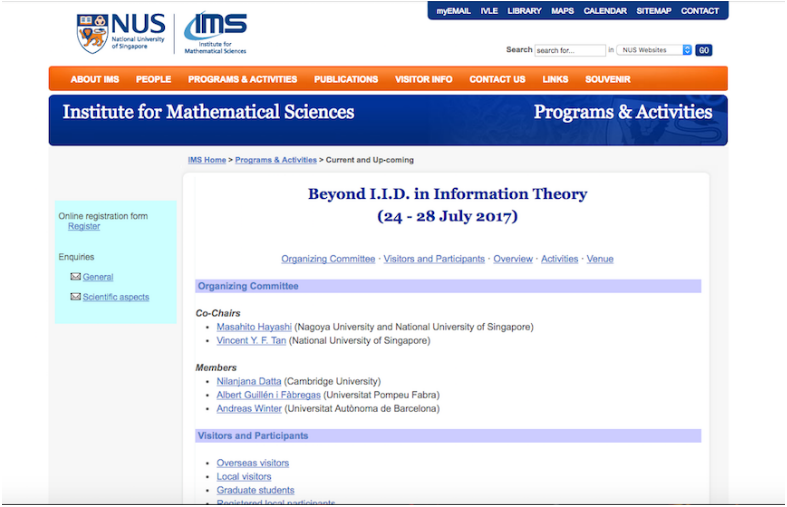
- Also provided a converse in the MD regime under some conditions
- See arXiv 1512.06298 for achievability and arXiv 1604.06848 for the converse

Invitation to Beyond IID 2017



Beyond IID 2017 will be at NUS, Singapore