The Third-Order Term in the Normal Approximation for the AWGN Channel

Vincent Y. F. Tan

Department of Electrical and Computer Engineering,
Department of Mathematics,
National University of Singapore

Joint work with Marco Tomamichel (CQT, NUS)

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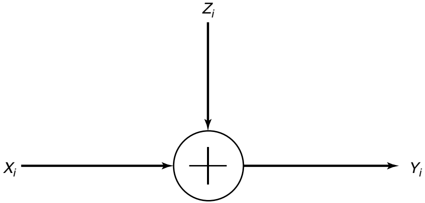
The AWGN Channel



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$$W(y|x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(y-x)^2}{2}\right)$$

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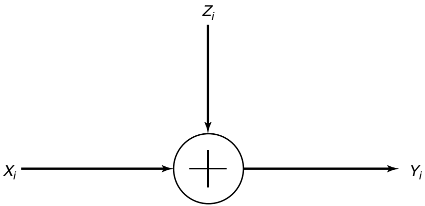

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- Assuming an average power constraint

$$\frac{1}{n}\sum_{i=1}^{n} X_i^2 \le P$$

  the capacity is the familiar expression

$$\mathsf{C}(P) = \frac{1}{2}\log(1+P) \qquad \text{bits per ch use}$$

Non-Asymptotic Definition and Strong Converse

- Let

$$M^*(n, \varepsilon, P) := \max\{m \in \mathbb{N} : \exists\,(n, m, \varepsilon, P)\text{–code}\}$$

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- From Shannon's result and the strong converse (e.g., Shannon (1959), Yoshihara (1964))
$$\lim_{n\to\infty} \frac{1}{n} \log M^*(n,\varepsilon,P) = \mathrm{C}(P), \qquad \forall\,\varepsilon \in (0,1)$$

Second-Order Asymptotics

- Hayashi (2009) and Polyanskiy-Poor-Verdú (2010) showed the more refined expansion

$$\log M^*(n, \varepsilon, P) = n\mathrm{C}(P) + \sqrt{n\mathrm{V}(P)}\Phi^{-1}(\varepsilon) + \theta_n$$

where $\mathrm{V}(P)$ is the Gaussian dispersion function defined as

$$\mathrm{V}(P) := \log^2 \mathrm{e} \cdot \frac{P(P+2)}{2(P+1)^2}, \qquad \text{squared bits per ch use}$$

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- Our main contribution is an improvement of the lower bound

Tight Third-Order Asymptotics

Theorem (Tan-Tomamichel (2013))

For all $P > 0$ *and* $\varepsilon \in (0,1)$*, we have*

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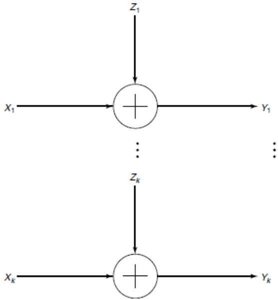
- Converse follows from an application of the hypothesis testing converse by Polyanskiy-Poor-Verdú (2010) or Hayashi-Nagaoka (2003) converse with output distribution

$$Q_{Y^n} = \mathcal{N}(0, P+1)^{\otimes n}$$

Extensions: Parallel Gaussian Channels



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Sum of powers equals P

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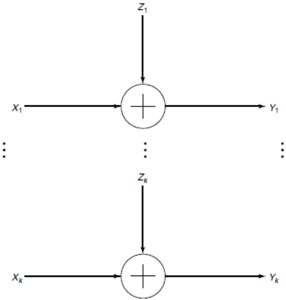

$$Z_{j,i} \stackrel{\text{iid}}{\sim} \mathcal{N}(0, N_j), \quad j = 1, \ldots, k$$

Sum of powers equals P

$$\log M^*(n, \varepsilon, P) \ge n \sum_{j=1}^{k} \mathrm{C}\left(\frac{P_j}{N_j}\right) + \sqrt{n \sum_{j=1}^{k} \mathrm{V}\left(\frac{P_j}{N_j}\right)} \Phi^{-1}(\varepsilon) + \frac{1}{2} \log n + O(1)$$

where $P_j = |\nu - N_j|^+$ and ν satisfies $\sum_{j=1}^{k} P_j = P$. Not third-order tight...

Relation to Prefactors for Error Exponents

- For high rates (rates above critical rate), it can be shown following Shannon (1959) that

$$\varepsilon^*(\lfloor \exp(nR) \rfloor, n) = \Theta\left(\frac{\exp(-nE(R))}{n^{(1+|E'(R)|)/2}} \right)$$

  where $E(R)$ is the reliability function of the AWGN channel and $E'(R) \le 0$ is the derivative.

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- Some similarities to third-order terms

Comparison of Prefactors to Third-Order Terms

Channel	Third-Order Term	EE Prefactor
AWGN (This Work)	$\frac{1}{2}\log n + O(1)$	$\frac{1}{n^{(1+\lvert E'(R)\rvert)/2}}$
Non-singular, Symmetric$^{\heartsuit}$ DMC	$\frac{1}{2}\log n + O(1)$ $\diamondsuit$	$\frac{1}{n^{(1+\lvert E'(R)\rvert)/2}}$ $\clubsuit$
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- $^\clubsuit$ Altuğ-Wagner (2011-2012), Scarlett *et al.* (2013)
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Proof Strategy

- We want to prove that

$$\log M^*(n,\varepsilon,P) \ge n\mathrm{C}(P) + \sqrt{n\mathrm{V}(P)}\Phi^{-1}(\varepsilon) + \frac{1}{2}\log n + \underline{K}(\varepsilon,P)$$

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5. Evaluation of RCU using Berry-Esseen

RCU Bound and Choice of Input Distribution

- RCU bound: For any input distribution P_{X^n} and decoding metric $q(x^n, y^n)$, there exists an (n, M, ε', P)-code satisfying

$$\varepsilon' \le \mathbb{E}\left[\min\left\{1, M \Pr\left(q(\bar{X}^n, Y^n) \ge q(X^n, Y^n) | X^n, Y^n\right)\right\}\right]$$

where $(X^n, \bar{X}^n, Y^n) \sim P_{X^n}(x^n) \times P_{X^n}(\bar{x}^n) \times W^n(y^n|x^n)$

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- We choose

$$P_{X^n}(x^n) \propto \delta\left(\|x^n\|_2^2 - nP\right)$$

  the uniform distribution on the power sphere $\{x^n : \|x^n\|_2^2 = nP\}$

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- Satisfies power constraints

$$\frac{1}{n}\sum_{i=1}^{n} X_i^2 \le P$$

  with probability one

Decoding Metric

- The decoding metric $q(x^n, y^n)$ is chosen as

$$q(x^n, y^n) := \log \frac{W^n(y^n|x^n)}{P_{X^n}W^n(y^n)}.$$

where $P_{X^n}W^n$ is the output distribution induced by P_{X^n} and W^n

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- Let the probability within the RCU bound be parametrized as

$$g(t, y^n) := \Pr\left(q(\bar{X}^n, Y^n) \geq t | Y^n = y^n\right)$$

  then it can be seen by using the definition of q and Bayes rule that

$$g(t, y^n) = \mathbb{E}\left[\exp(-q(X^n, Y^n))\mathbb{I}\{q(X^n, Y^n) \geq t\} \,\middle|\, Y^n = y^n\right]$$

- It is imperative to understand the behavior of $q(X^n, Y^n)$

Inner Product and Typical Channel Outputs

- By standard manipulations, we have

$$q(x^n, y^n) = \frac{n}{2}\log\frac{1}{2\pi} + \langle x^n, y^n\rangle - nP - \|y^n\|_2^2 - \log P_{X^n}W^n(y^n)$$

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  where $(X^n, Y^n) \sim P_{X^n} \times W^n$ is not a sequence of iid random variables

- We may assume that Y^n is typical in the sense that

$$\frac{1}{n}\|Y^n\|_2^2 \in [P + 1 - \delta_n, P + 1 + \delta_n]$$

  and $\delta_n = n^{-1/3}$

Probability of Log-Likelihood in An Interval

Lemma

For y^n typical, the following holds for any a and μ

$$\Pr\left(q(X^n, Y^n) \in [a, a+\mu] \,\middle|\, Y^n = y^n\right) \le \kappa(P) \cdot \frac{\mu}{\sqrt{n}}.$$

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- Say $\|y^n\|_2^2 = ns$ where $s \in [P+1-\delta_n, P+1+\delta_n]$. Then we simply have to consider

$$\Pr\left(\langle X^n, Z^n\rangle \in [b, b+\mu] \,\middle|\, \|X^n + Z^n\|_2^2 = ns\right)$$

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- By spherical symmetry, choose $X^n = x_0^n := (\sqrt{nP}, 0, \ldots, 0)$ so

$$\Pr\left(Z_1 + \sqrt{nP} \in \left[\frac{b}{\sqrt{nP}}, \frac{b+\mu}{\sqrt{nP}}\right] \,\middle|\, \|x_0^n + Z^n\|_2^2 = ns\right)$$

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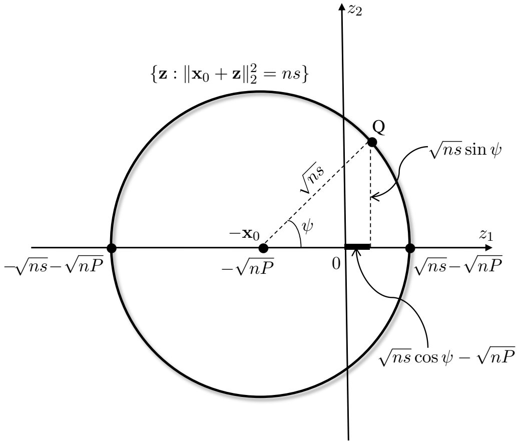

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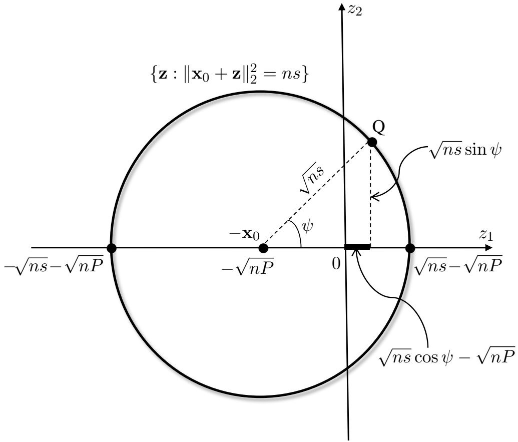

Probability that $Z_1 + \sqrt{nP}$ belongs to an interval of length $\mu/\sqrt{n}$ if Z^n lands on the sphere with radius $\sqrt{ns}$ centered at $(-\sqrt{nP}, 0, \ldots, 0)$?

Probability of Log-Likelihood in An Interval

- Leverage on radial symmetry
- Change coordinates

$$Z_1 = \sqrt{ns}\cos\Psi - \sqrt{nP}$$

- Apply Laplace approximation for integrals to the conditional probability density of Ψ given Z^n lands on sphere to prove lemma, i.e.,

$$\Pr\left(Z_1 + \sqrt{nP} \in \left[\frac{b}{\sqrt{nP}}, \frac{b+\mu}{\sqrt{nP}}\right] \,\middle|\, \|x_0^n + Z^n\|_2^2 = ns\right) \le O\left(\frac{\mu}{\sqrt{n}}\right)$$

Probability of Decoding Metric Exceeding t

- Recall that

$$g(t, y^n) = \Pr(q(\bar{X}^n, Y^n) \geq t \,|\, Y^n = y^n)$$

- Using the Lemma, we can upper bound $g(t, y^n)$ (uniformly for typical y^n) as

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- The $\sqrt{n}$ above contributes to the achievability of the $\frac{1}{2}\log n$ term

Conclusion

- We completed the story up to the third-order for AWGN channels

$$\log M^*(n, \varepsilon, P) = n\mathrm{C}(P) + \sqrt{n\mathrm{V}(P)}\Phi^{-1}(\varepsilon) + \frac{1}{2}\log n + O(1)$$

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- For detailed derivations, see http://arxiv.org/abs/1311.2337