Rank Minimization over Finite Fields: Fundamental Limits and Coding-Theoretic Interpretations

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Joint work with

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Stark C. Draper

Outline

1. Motivation

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2. Problem Setup and Summary of Main Results

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3. Converse

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6. Coding-Theoretic Interpretations

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Connection between Two Areas of Study

Area of Study	Matrix Completion Rank Minimization	Rank-Metric Codes

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The *rank distance* *between matrices* $\mathbf{A}, \mathbf{B} \in \mathbb{F}_q^{m \times n}$ *is defined as*

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A rank-metric code is $\mathcal{C}$ is a non-empty subset of $\mathbb{F}_q^{m \times n}$ endowed with the rank distance.

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- Transmit some codeword $\mathbf{C} \in \mathcal{C}$

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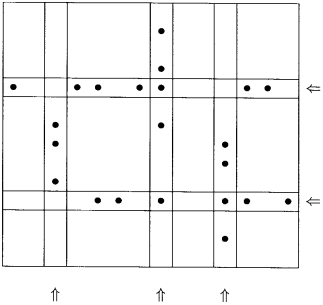
Roth, IT Trans 1997



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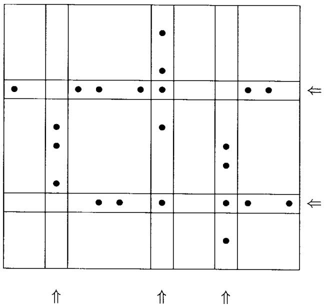

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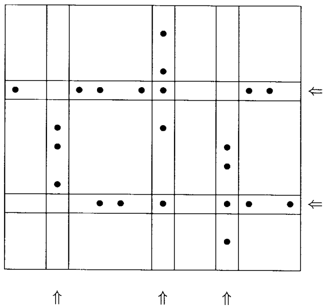

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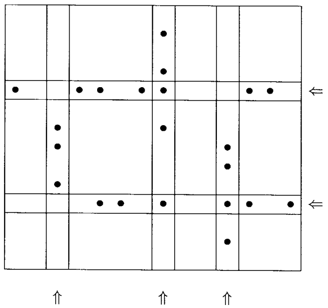

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- Side information $\{x_j : (i,j) \in \mathcal{G}\}$

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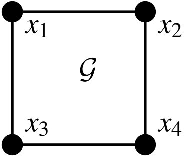

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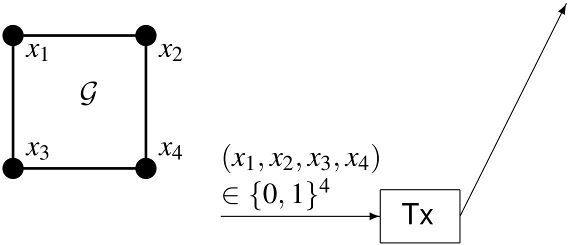

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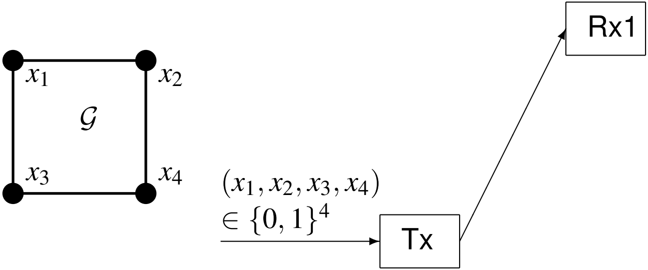

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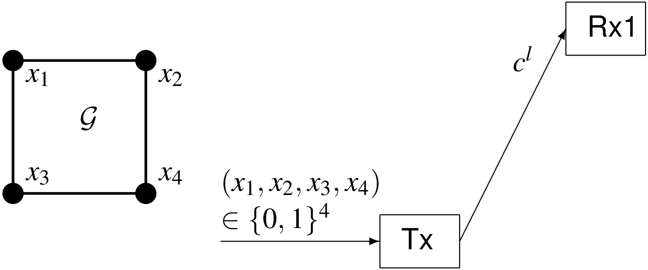

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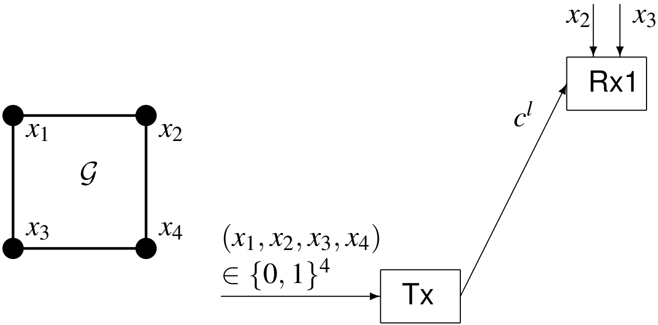

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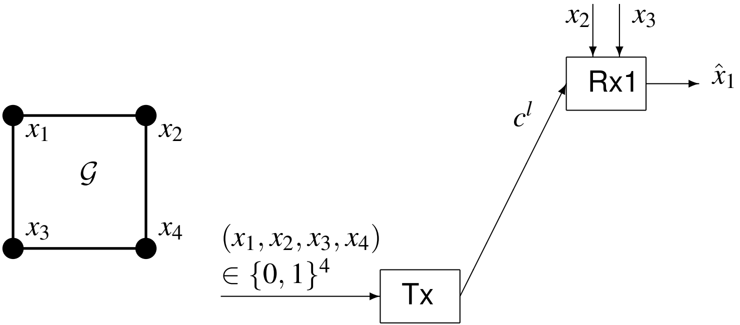

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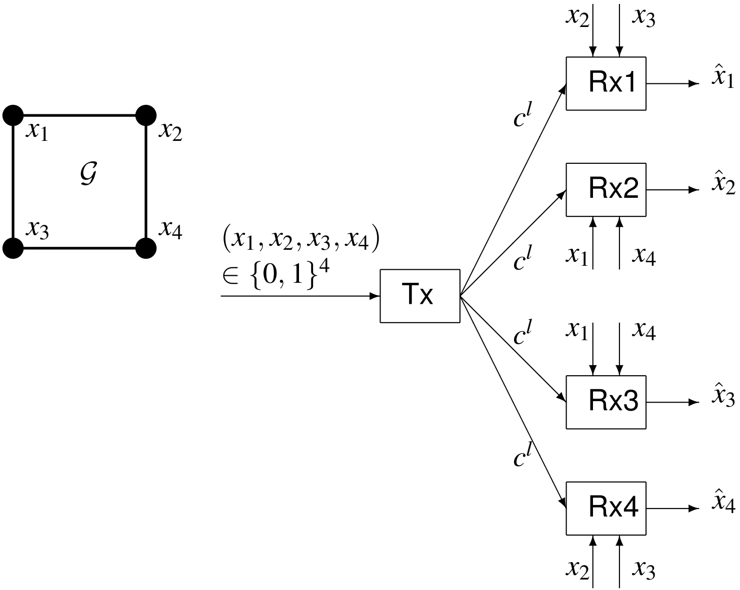

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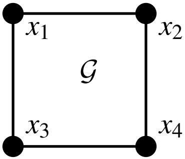

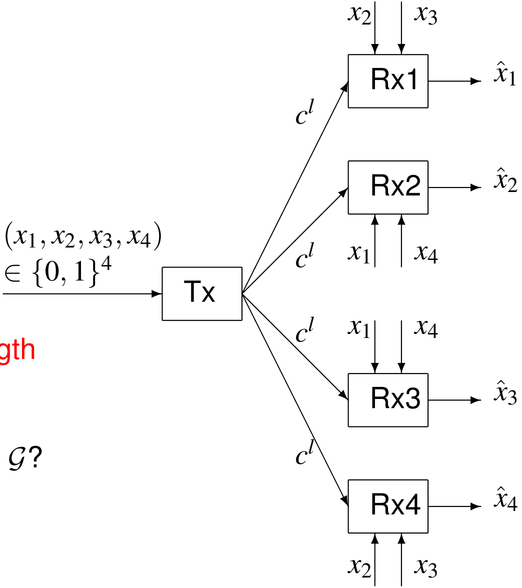

What's the minimum length l required for reliable reconstruction of (x_1, x_2, x_3, x_4) given graph $\mathcal{G}$?

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We say that a matrix $\mathbf{A} \in \mathbb{F}_2^{k \times k}$ fits a graph $\mathcal{G}$ if

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Theorem (Bar-Yossef et al. 06)

The optimal length of a linear index code for graph $\mathcal{G}$ equals $\mathrm{minrk}_2(\mathcal{G})$.

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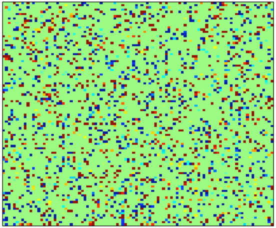

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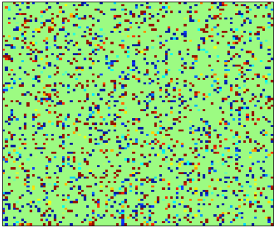

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- The nuclear norm heuristic has also enjoyed tremendous successes
- Wide applicability in for e.g., collaborative filtering, minimal realization of LTI system etc.

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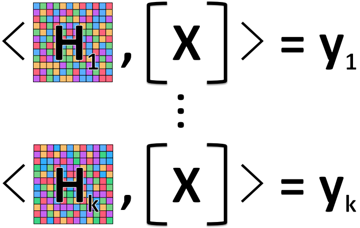

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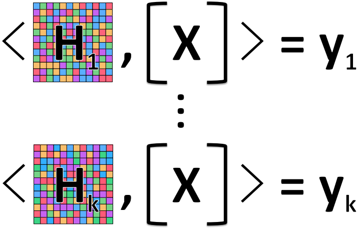

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- Each measurement $y_a, a = 1, \ldots, k$ belongs to $\mathbb{F}_q$

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- Find necessary and sufficient conditions on k and sensing model such that recovery is reliable, i.e.,

$$\mathbb{P}(\mathcal{E}_n) \to 0$$

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- Coding-theoretic interpretations: Geometric insights

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Achievability (Uniform)	$k > (2+\varepsilon)\gamma(1-\gamma/2)n^2$	$\mathbb{P}(\mathcal{E}_n) \to 0$ $\mathbb{P}(\mathcal{E}_n) \approx q^{-n^2 E(R)}$

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Achievability (Sparse)	$k > (2+\varepsilon)\gamma(1-\gamma/2)n^2$	$\mathbb{P}(\mathcal{E}_n) \to 0$

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Achievability (Sparse)	$k > (2+\varepsilon)\gamma(1-\gamma/2)n^2$	$\mathbb{P}(\mathcal{E}_n) \to 0$
Achievability (Noisy)	$k \gtrapprox (3+\varepsilon)(\gamma+\sigma)n^2$ (q assumed large)	$\mathbb{P}(\mathcal{E}_n) \to 0$

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A necessary condition on number of measurements

Given k measurements $y_a \in \mathbb{F}_q$ and sensing matrices $\mathbf{H}_a \in \mathbb{F}_q^{n \times n}$, we want a necessary condition for reliable recovery of $\mathbf{X}$.

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If the number of measurements satisfies

$$k < (2-\varepsilon)\gamma\left(1-\frac{\gamma}{2}\right)n^2$$

then $\mathbb{P}(\hat{\mathbf{X}} \neq \mathbf{X}) \geq \varepsilon/2$ *for all* n *sufficiently large.*

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Outline

1. Motivation
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3. Converse
4. Achievability for Uniform Model
5. Achievability for Sparse Model
6. Coding-Theoretic Interpretations

A sensing model: Uniform model

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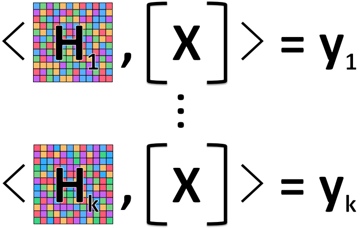

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- We want the solution to be unique and correct

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By the union bound, the error probability can be bounded as

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That the number of matrices of rank $\leq r$ is bounded above as

$$4q^{2\gamma(1-\gamma/2)n^2}$$

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The rate of a sequence of linear measurement models is defined as

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Analogy to coding: The rate of the code

$$\mathcal{C} = \{\mathbf{C} : \langle \mathbf{C}, \mathbf{H}_a \rangle = 0, a = 1, \ldots, k\}$$

is lower bounded by $1 - k/n^2$.

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Reliability function also known as error exponent

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Then,

$$E(R) = \left|(1-R) - 2\gamma\left(1 - \frac{\gamma}{2}\right)\right|^{+}$$

Note $|x|^{+} := \max\{x, 0\}$.

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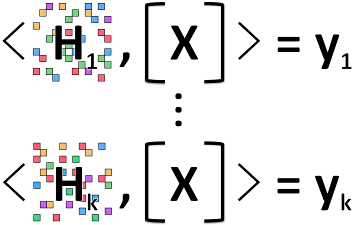

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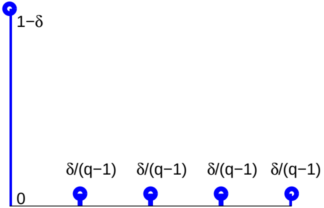

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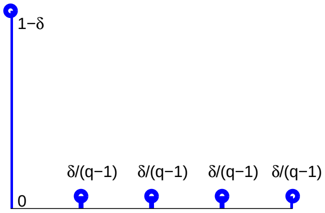

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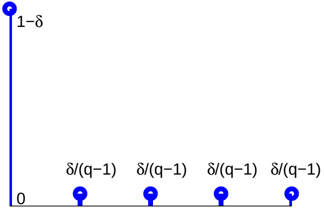

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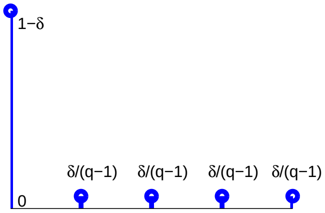

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- How fast can δ, the sparsity factor, decay with n for reliable recovery?

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Assume

- *Sensing matrices* $\mathbf{H}_a$ *drawn according to* δ*-sparse distribution*
- *Min-rank decoder is used*
- $r/n \to \gamma$ *(constant)*

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If the sparsity factor satisfies

$$\delta \in \Omega\left(\frac{\log n}{n}\right)$$

and the number of measurements satisfies

$$k > (2+\varepsilon)\gamma\left(1-\frac{\gamma}{2}\right)n^2$$

then $\mathbb{P}(\mathcal{E}_n) \to 0$.

Remarks on achievability under sparse model

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- Average number of entries in $\mathbf{H}_a$ is $\Omega(n \log n)$
- Number of measurements matches uniform model and information-theoretic (Fano's) lower bound
- Roughly speaking, if

$$\delta \in \Omega\left(\frac{\log n}{n}\right)$$

probability of error events still "look like" the uniform model

Proof Strategy

Approximate with uniform model

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$$\mathbb{P}(\mathcal{E}_n) \le \sum_{\substack{\mathbf{Z}\neq\mathbf{X}:\mathrm{rank}(\mathbf{Z})\le\mathrm{rank}(\mathbf{X})\\ \|\mathbf{Z}-\mathbf{X}\|_0\le\beta n^2}} \mathbb{P}(\langle\mathbf{Z},\mathbf{H}_a\rangle = \langle\mathbf{X},\mathbf{H}_a\rangle, \forall a = 1,\ldots,k)$$

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- First term
  - Few terms
  - Low Hamming weight – can afford loose bound on probability
- Second term
  - Many terms
  - Tight bound on probability (circular convolution of δ-sparse pmf)

Outline

1 Motivation

2 Problem Setup and Summary of Main Results

3 Converse

4 Achievability for Uniform Model

5 Achievability for Sparse Model

6 Coding-Theoretic Interpretations

Interpreting our results from the rank-metric lens

- Recall that a rank-metric code $\mathcal{C}$ is a non-empty subset of $\mathbb{F}_q^{m \times n}$ endowed with the rank distance

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$$\begin{aligned} &\text{minimize} \quad && \mathrm{rank}(\mathbf{X}) \\ &\text{subject to} \quad && \langle \mathbf{H}_a, \mathbf{X}\rangle = y_a, \quad a = 1,\ldots,k \end{aligned}$$

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with the identification $\mathbf{X} \equiv \mathbf{R}-\mathbf{C}$ and

$$\mathcal{C} := \{\mathbf{C} : \langle \mathbf{C}, \mathbf{H}_a\rangle = 0, a = 1,\ldots,k\}$$

Minimum rank distance

Definition

The *minimum rank distance* *of a rank-metric code* $\mathcal{C}$ *is*

$$d(\mathcal{C}) := \min_{\mathbf{C}_1, \mathbf{C}_2 \in \mathcal{C} : \mathbf{C}_1 \neq \mathbf{C}_2} \operatorname{rank}(\mathbf{C}_1 - \mathbf{C}_2)$$

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If the code is linear (as it is)

$$d(\mathcal{C}) := \min_{\mathbf{C} \in \mathcal{C} : \mathbf{C} \neq \mathbf{0}} \operatorname{rank}(\mathbf{C})$$

How does $d(\mathcal{C})$ behave for the random linear rank-metric code under the uniform and sparse models?

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The rate of the code is defined as

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Proposition (Concentration of minimum rank distance)

For the uniform model, with probability approaching 1 as $n \to \infty$, the relative minimum distance satisfies

$$\frac{d(\mathcal{C})}{n} \in (1 - \sqrt{R} - \varepsilon, 1 - \sqrt{R} + \varepsilon)$$

Minimum rank distance: Interpretation

- The rate-dependent function

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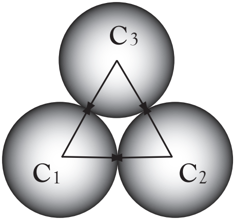
Uniform rank-metric code has minimum distance

$$n\gamma_{\mathsf{GV}}(R) = n(1 - \sqrt{R})$$

with high probability for n large



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For successful recovery of $\mathbf{X}$ on average,

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Achievability result may be derived by setting $R = 1 - \frac{k}{n^2}$ and considering

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which is equivalent to

$$k \geq (2 + \varepsilon')\gamma \left(1 - \frac{\gamma}{2}\right) n^2$$

Minimum distances for the sparse model

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- The minimum distance properties of the uniform and sparse models are identical
- So are their measurement complexities!

$$k > (2+\varepsilon)\gamma\left(1 - \frac{\gamma}{2}\right)n^2$$

Concluding remarks

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- Derived minimum distance properties of rank-metric codes
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- Reduced complexity of the min-rank decoder vis-à-vis exhaustive search

Open Problems

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Check out the preprint: `http://arxiv.org/abs/1104.4302`