Improved Bounds on Sidon Sets via Lattice Packing of Simplices

Vincent Y. F. Tan

(joint work with Mladen Kovačević)

ITA (February 15, 2018)

Sidon Sets

Definition

A subset $B = \{b_0, b_1, \ldots, b_n\}$ of a finite Abelian group G is called a Sidon set of order h if the sums $b_{i_1} + \cdots + b_{i_h}$ are distinct for every choice of $0 \leq i_1 \leq \cdots \leq i_h \leq n$.

Sidon Sets

Definition

A subset $B = \{b_0, b_1, \ldots, b_n\}$ of a finite Abelian group G is called a Sidon set of order h if the sums $b_{i_1} + \cdots + b_{i_h}$ are distinct for every choice of $0 \leq i_1 \leq \cdots \leq i_h \leq n$.

- If B is a Sidon set, then so is its translate

$$B - b_0 = \{0, b_1 - b_0, \ldots, b_n - b_0\},$$

  and vice versa.

  $\Rightarrow$ We can assume w.l.o.g. that $b_0 = 0$

- With this convention, B is a Sidon set if and only if the sums $b_{i_1} + \cdots + b_{i_t}$ are distinct for every choice of $1 \leq i_1 \leq \cdots \leq i_t \leq n$ and $0 \leq t \leq h$.

Sidon Sets: The Case $h = 2$

Sidon Sets: The Case $h = 2$

- The requirement here is that all pairwise sums $b_{i_1} + b_{i_2}$ are distinct (up to the order of the summands)

Sidon Sets: The Case $h = 2$

- The requirement here is that all pairwise sums $b_{i_1} + b_{i_2}$ are distinct (up to the order of the summands)

Example

Integers modulo 13*:*

$$0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \quad 8 \quad 9 \quad 10 \quad 11 \quad 12$$

Sidon set $\{0, 1, 3, 9\} \subset \mathbb{Z}_{13}$.

Sidon Sets: The Case $h = 2$

- The requirement here is that all pairwise sums $b_{i_1} + b_{i_2}$ are distinct (up to the order of the summands)

Example

Integers modulo 13*:*

$$0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \quad 8 \quad 9 \quad 10 \quad 11 \quad 12$$

Sidon set $\{0, 1, 3, 9\} \subset \mathbb{Z}_{13}$.

Sidon Sets: The Case $h = 2$

- The requirement here is that all pairwise sums $b_{i_1} + b_{i_2}$ are distinct (up to the order of the summands)

Example

Integers modulo 13*:*

0

Sidon set $\{0, 1, 3, 9\} \subset \mathbb{Z}_{13}$. $0 + 0 = 0$

Sidon Sets: The Case $h = 2$

- The requirement here is that all pairwise sums $b_{i_1} + b_{i_2}$ are distinct (up to the order of the summands)

Example

Integers modulo 13*:*

$0 \quad 1$

Sidon set $\{0, 1, 3, 9\} \subset \mathbb{Z}_{13}$.

$0 + 1 = 1$

Sidon Sets: The Case $h = 2$

- The requirement here is that all pairwise sums $b_{i_1} + b_{i_2}$ are distinct (up to the order of the summands)

Example

Integers modulo 13*:*

$0 \quad 1 \qquad 3$

Sidon set $\{0, 1, 3, 9\} \subset \mathbb{Z}_{13}$. $\quad 0 + 3 = 3$

Sidon Sets: The Case $h = 2$

- The requirement here is that all pairwise sums $b_{i_1} + b_{i_2}$ are distinct (up to the order of the summands)

Example

Integers modulo 13*:*

$0 \quad 1 \quad\quad 3 \quad\quad\quad\quad\quad\quad 9$

Sidon set $\{0, 1, 3, 9\} \subset \mathbb{Z}_{13}$. $\quad\quad 0 + 9 = 9$

Sidon Sets: The Case $h = 2$

- The requirement here is that all pairwise sums $b_{i_1} + b_{i_2}$ are distinct (up to the order of the summands)

Example

Integers modulo 13*:*

$0 \quad 1 \quad 2 \quad 3 \qquad\qquad 9$

Sidon set $\{0, 1, 3, 9\} \subset \mathbb{Z}_{13}$. $\qquad 1 + 1 = 2$

Sidon Sets: The Case $h = 2$

- The requirement here is that all pairwise sums $b_{i_1} + b_{i_2}$ are distinct (up to the order of the summands)

Example

Integers modulo 13*:*

$0 \quad 1 \quad 2 \quad 3 \quad 4 \qquad\qquad\qquad\qquad 9$

Sidon set $\{0, 1, 3, 9\} \subset \mathbb{Z}_{13}$.

$1 + 3 = 4$

Sidon Sets: The Case $h = 2$

- The requirement here is that all pairwise sums $b_{i_1} + b_{i_2}$ are distinct (up to the order of the summands)

Example

Integers modulo 13*:*

$$0 \quad 1 \quad 2 \quad 3 \quad 4 \qquad\qquad\qquad 9 \quad 10$$

Sidon set $\{0, 1, 3, 9\} \subset \mathbb{Z}_{13}$. $\qquad 1 + 9 = 10$

Sidon Sets: The Case $h = 2$

- The requirement here is that all pairwise sums $b_{i_1} + b_{i_2}$ are distinct (up to the order of the summands)

Example

Integers modulo 13*:*

$$0 \quad 1 \quad 2 \quad 3 \quad 4 \qquad 6 \qquad\qquad 9 \quad 10$$

Sidon set $\{0, 1, 3, 9\} \subset \mathbb{Z}_{13}$. $3 + 3 = 6$

Sidon Sets: The Case $h = 2$

- The requirement here is that all pairwise sums $b_{i_1} + b_{i_2}$ are distinct (up to the order of the summands)

Example

Integers modulo 13*:*

$$0 \quad 1 \quad 2 \quad 3 \quad 4 \qquad 6 \qquad\qquad 9 \quad 10 \qquad {\color{red}12}$$

Sidon set $\{0, 1, 3, 9\} \subset \mathbb{Z}_{13}$. $\quad {\color{red}3 + 9 = 12}$

Sidon Sets: The Case $h = 2$

- The requirement here is that all pairwise sums $b_{i_1} + b_{i_2}$ are distinct (up to the order of the summands)

Example

Integers modulo 13*:*

$0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \qquad\qquad 9 \quad 10 \qquad\qquad 12$

Sidon set $\{0, 1, 3, 9\} \subset \mathbb{Z}_{13}$.

$9 + 9 = 18 \equiv 5$

Sidon Sets: The Case $h = 2$

- The requirement here is that all pairwise sums $b_{i_1} + b_{i_2}$ are distinct (up to the order of the summands)

Example

Integers modulo 13*:*

$$0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \qquad\qquad\qquad 9 \quad 10 \qquad\qquad 12$$

Sidon set $\{0, 1, 3, 9\} \subset \mathbb{Z}_{13}$.

Constructions of Sidon Sets

Constructions of Sidon Sets

Theorem (Singer '38)

There exists a Sidon set of order $h = 2$ and cardinality $n + 1$ in the group $\mathbb{Z}_{n^2+n+1}$, whenever n is a prime power.

Constructions of Sidon Sets

Theorem (Singer '38)

There exists a Sidon set of order $h = 2$ *and cardinality* $n + 1$ *in the group* $\mathbb{Z}_{n^2+n+1}$*, whenever* n *is a prime power.*

Theorem (Bose–Chowla '62)

(a) *There exists a Sidon set of order* h *and cardinality* $n + 1$ *in the group* $\mathbb{Z}_{n^h+n^{h-1}+\cdots+1}$*, whenever* n *is a prime power.*

Constructions of Sidon Sets

Theorem (Singer '38)

There exists a Sidon set of order $h = 2$ *and cardinality* $n + 1$ *in the group* $\mathbb{Z}_{n^2+n+1}$*, whenever* n *is a prime power.*

Theorem (Bose–Chowla '62)

(a) *There exists a Sidon set of order* h *and cardinality* $n + 1$ *in the group* $\mathbb{Z}_{n^h+n^{h-1}+\cdots+1}$*, whenever* n *is a prime power.*

(b) *There exists a Sidon set of order* h *and cardinality* $n + 1$ *in the group* $\mathbb{Z}_{(n+1)^h-1}$*, whenever* $n + 1$ *is a prime power.*

Constructions of Sidon Sets

Theorem (Singer '38)

There exists a Sidon set of order $h = 2$ and cardinality $n + 1$ in the group $\mathbb{Z}_{n^2+n+1}$, whenever n is a prime power.

Theorem (Bose–Chowla '62)

(a) *There exists a Sidon set of order h and cardinality $n + 1$ in the group $\mathbb{Z}_{n^h+n^{h-1}+\cdots+1}$, whenever n is a prime power.*

(b) *There exists a Sidon set of order h and cardinality $n + 1$ in the group $\mathbb{Z}_{(n+1)^h-1}$, whenever $n + 1$ is a prime power.*

- For h fixed and $n \to \infty$, it is conjectured that the optimal size of the group grows as $\sim n^h$

Constructions of Sidon Sets

Theorem (Singer '38)

There exists a Sidon set of order $h = 2$ and cardinality $n + 1$ in the group $\mathbb{Z}_{n^2+n+1}$, whenever n is a prime power.

Theorem (Bose–Chowla '62)

(a) *There exists a Sidon set of order h and cardinality $n + 1$ in the group $\mathbb{Z}_{n^h+n^{h-1}+\cdots+1}$, whenever n is a prime power.*

(b) *There exists a Sidon set of order h and cardinality $n + 1$ in the group $\mathbb{Z}_{(n+1)^h-1}$, whenever $n + 1$ is a prime power.*

- For h fixed and $n \to \infty$, it is conjectured that the optimal size of the group grows as $\sim n^h$
- This is true for $h = 2$ from Singer's construction.

Packings of Simplices

Packings of Simplices

- The discrete simplex is the following set in $\mathbb{Z}^n$:

$$\triangle_h^n = \left\{ \mathbf{y} \in \mathbb{Z}^n \ : \ y_i \geq 0, \ \sum_{i=1}^{n} y_i \leq h \right\}$$

Packings of Simplices

- The discrete simplex is the following set in $\mathbb{Z}^n$:

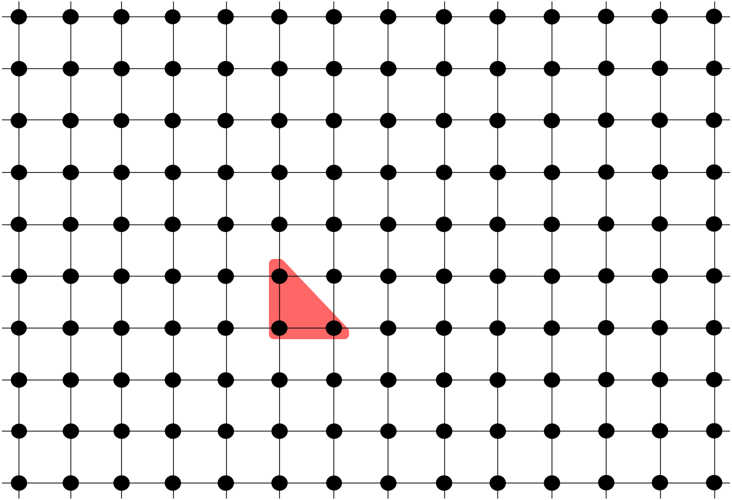
$$\triangle_h^n = \left\{ \mathbf{y} \in \mathbb{Z}^n \;:\; y_i \geq 0,\; \sum_{i=1}^{n} y_i \leq h \right\}$$

- Its cardinality is

$$|\triangle_h^n| = \binom{h+n}{n} \sim \frac{h^n}{n!} \quad \text{as} \quad h \to \infty.$$

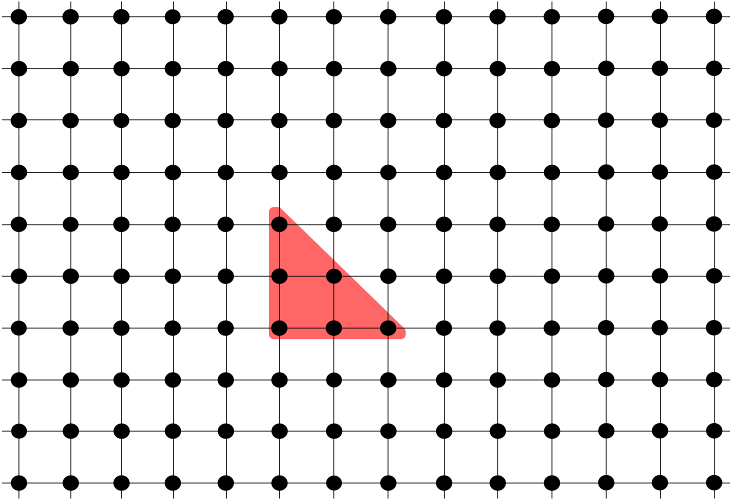
Packings of Simplices

- The simplex $\triangle_1^2$



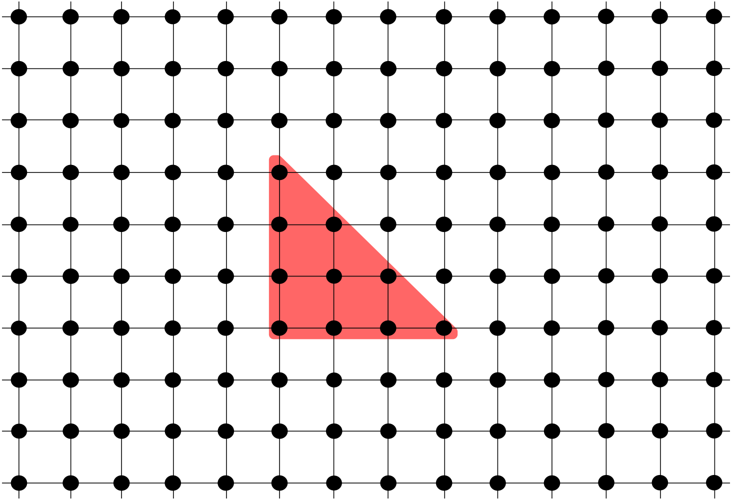
Packings of Simplices

- The simplex $\triangle_2^2$



Packings of Simplices

- The simplex $\triangle_3^2$



Packings of Simplices

Packings of Simplices

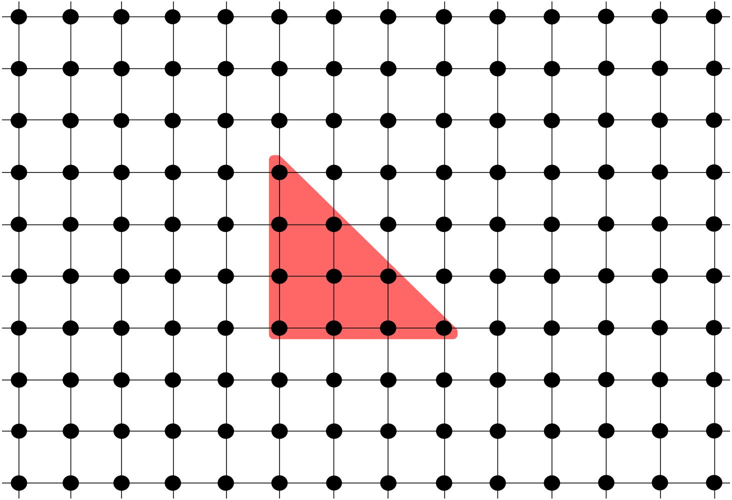
- Let $\mathcal{T} \subseteq \mathbb{Z}^n$. We say that $(\triangle_h^n, \mathcal{T})$ is a packing in $\mathbb{Z}^n$ if the translates $\mathbf{x} + \triangle_h^n$ and $\mathbf{x}' + \triangle_h^n$ are disjoint for every $\mathbf{x}, \mathbf{x}' \in \mathcal{T}$, $\mathbf{x} \neq \mathbf{x}'$

Packings of Simplices

- Let $\mathcal{T} \subseteq \mathbb{Z}^n$. We say that $(\triangle_h^n, \mathcal{T})$ is a packing in $\mathbb{Z}^n$ if the translates $\mathbf{x} + \triangle_h^n$ and $\mathbf{x}' + \triangle_h^n$ are disjoint for every $\mathbf{x}, \mathbf{x}' \in \mathcal{T}$, $\mathbf{x} \neq \mathbf{x}'$
- If $\mathcal{T}$ is a lattice (a subgroup of $\mathbb{Z}^n$), such a packing is called a lattice packing

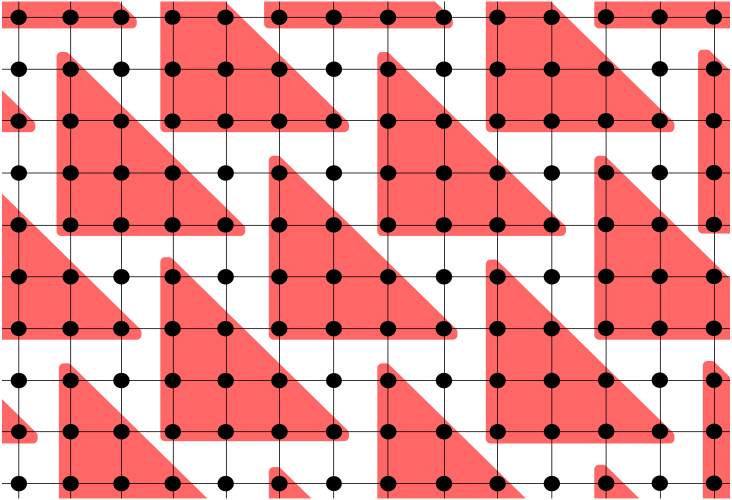
Packings of Simplices

- The simplex $\triangle_3^2$



Packings of Simplices

- Lattice packing of the simplex $\triangle_3^2$



Geometry of Sidon Sets

Theorem

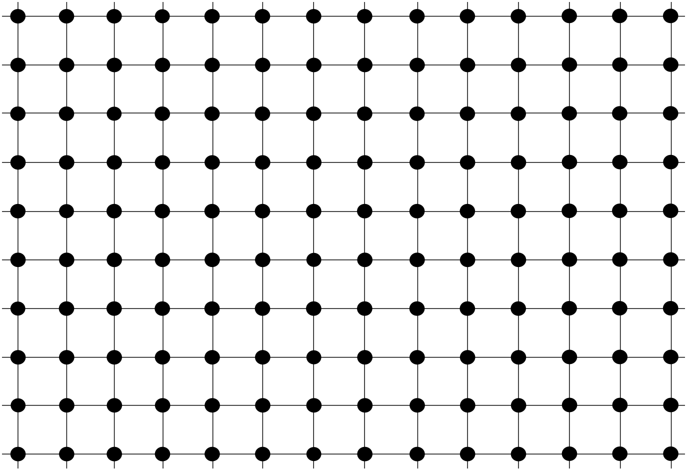
(a) *If* $B = \{0, b_1, \ldots, b_n\}$ *is a Sidon set of order* h *in an Abelian group* G*, then* $(\triangle_h^n, \mathcal{L})$ *is a lattice packing in* $\mathbb{Z}^n$*, where*

$$\mathcal{L} = \left\{ \mathbf{x} \in \mathbb{Z}^n : \sum_{i=1}^{n} x_i \cdot b_i = 0 \right\}.$$

If, in addition, B *generates* G*, then* $G \cong \mathbb{Z}^n / \mathcal{L}$*.*

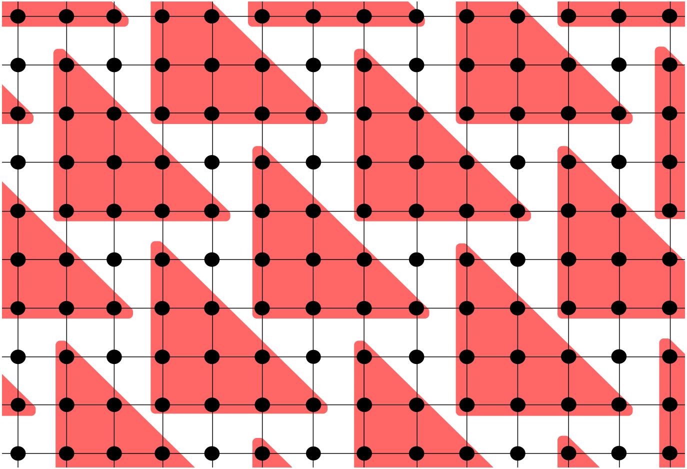
Geometry of Sidon Sets: Example

- The packing $(\triangle_3^2, \mathcal{L})$ in $\mathbb{Z}^2$ that corresponds to the Sidon set $\{(0,0),(1,1),(0,5)\} \subset \mathbb{Z}_2 \times \mathbb{Z}_6$ of order $h = 3$



Geometry of Sidon Sets: Example

- The packing $(\triangle_3^2, \mathcal{L})$ in $\mathbb{Z}^2$ that corresponds to the Sidon set $\{(0,0),(1,1),(0,5)\} \subset \mathbb{Z}_2 \times \mathbb{Z}_6$ of order $h = 3$



Geometry of Sidon Sets

Theorem

(a) *If* $B = \{0, b_1, \ldots, b_n\}$ *is a Sidon set of order* h *in an Abelian group* G*, then* $(\triangle_h^n, \mathcal{L})$ *is a lattice packing in* $\mathbb{Z}^n$*, where*

$$\mathcal{L} = \left\{ \mathbf{x} \in \mathbb{Z}^n : \sum_{i=1}^{n} x_i \cdot b_i = 0 \right\}.$$

If, in addition, B *generates* G*, then* $G \cong \mathbb{Z}^n / \mathcal{L}$*.*

Geometry of Sidon Sets

Theorem

(a) *If* $B = \{0, b_1, \ldots, b_n\}$ *is a Sidon set of order* h *in an Abelian group* G*, then* $(\triangle_h^n, \mathcal{L})$ *is a lattice packing in* $\mathbb{Z}^n$*, where*

$$\mathcal{L} = \left\{ \mathbf{x} \in \mathbb{Z}^n : \sum_{i=1}^{n} x_i \cdot b_i = 0 \right\}.$$

If, in addition, B *generates* G*, then* $G \cong \mathbb{Z}^n / \mathcal{L}$*.*

(b) *Conversely, if* $(\triangle_h^n, \mathcal{L}')$ *is a lattice packing in* $\mathbb{Z}^n$*, then the group* $G = \mathbb{Z}^n / \mathcal{L}'$ *contains a Sidon set of order* h *and cardinality* $n + 1$*.*

Geometry of Sidon Sets

Theorem

(a) *If* $B = \{0, b_1, \ldots, b_n\}$ *is a Sidon set of order* h *in an Abelian group* G*, then* $(\triangle_h^n, \mathcal{L})$ *is a lattice packing in* $\mathbb{Z}^n$*, where*

$$\mathcal{L} = \left\{ \mathbf{x} \in \mathbb{Z}^n : \sum_{i=1}^{n} x_i \cdot b_i = 0 \right\}.$$

If, in addition, B *generates* G*, then* $G \cong \mathbb{Z}^n / \mathcal{L}$*.*

(b) *Conversely, if* $(\triangle_h^n, \mathcal{L}')$ *is a lattice packing in* $\mathbb{Z}^n$*, then the group* $G = \mathbb{Z}^n / \mathcal{L}'$ *contains a Sidon set of order* h *and cardinality* $n + 1$*.*

$\Rightarrow$ Lattice packings of simplices are geometric equivalents of Sidon sets in finite Abelian groups

Proof of (a):

- Suppose that $(\triangle_h^n, \mathcal{L})$ is not a packing, i.e., that the translates $\mathbf{x} + \triangle_h^n$ and $\mathbf{x}' + \triangle_h^n$ overlap for some distinct $\mathbf{x}, \mathbf{x}' \in \mathcal{L}$

Proof of (a):

- Suppose that $(\triangle_h^n, \mathcal{L})$ is not a packing, i.e., that the translates $\mathbf{x} + \triangle_h^n$ and $\mathbf{x}' + \triangle_h^n$ overlap for some distinct $\mathbf{x}, \mathbf{x}' \in \mathcal{L}$
- This means that there is a point $\mathbf{y} \in \mathbb{Z}^n$ which can be expressed as $\mathbf{y} = \mathbf{x} + \mathbf{f} = \mathbf{x}' + \mathbf{f}'$, where $\mathbf{x}, \mathbf{x}' \in \mathcal{L}$ are two different lattice points, and $\mathbf{f}, \mathbf{f}'$ are two (necessarily) different vectors in the simplex $\triangle_h^n$

Proof of (a):

- Suppose that $(\triangle_h^n, \mathcal{L})$ is not a packing, i.e., that the translates $\mathbf{x} + \triangle_h^n$ and $\mathbf{x}' + \triangle_h^n$ overlap for some distinct $\mathbf{x}, \mathbf{x}' \in \mathcal{L}$
- This means that there is a point $\mathbf{y} \in \mathbb{Z}^n$ which can be expressed as $\mathbf{y} = \mathbf{x} + \mathbf{f} = \mathbf{x}' + \mathbf{f}'$, where $\mathbf{x}, \mathbf{x}' \in \mathcal{L}$ are two different lattice points, and $\mathbf{f}, \mathbf{f}'$ are two (necessarily) different vectors in the simplex $\triangle_h^n$
- This implies that

$$\sum_{i=1}^n y_i \cdot b_i = \sum_{i=1}^n (x_i + f_i) \cdot b_i = \sum_{i=1}^n (x_i' + f_i') \cdot b_i$$

Proof of (a):

- Suppose that $(\triangle_h^n, \mathcal{L})$ is not a packing, i.e., that the translates $\mathbf{x} + \triangle_h^n$ and $\mathbf{x}' + \triangle_h^n$ overlap for some distinct $\mathbf{x}, \mathbf{x}' \in \mathcal{L}$
- This means that there is a point $\mathbf{y} \in \mathbb{Z}^n$ which can be expressed as $\mathbf{y} = \mathbf{x} + \mathbf{f} = \mathbf{x}' + \mathbf{f}'$, where $\mathbf{x}, \mathbf{x}' \in \mathcal{L}$ are two different lattice points, and $\mathbf{f}, \mathbf{f}'$ are two (necessarily) different vectors in the simplex $\triangle_h^n$
- This implies that

$$\sum_{i=1}^{n} y_i \cdot b_i = \sum_{i=1}^{n} (x_i + f_i) \cdot b_i = \sum_{i=1}^{n} (x_i' + f_i') \cdot b_i$$

- By definition of $\mathcal{L}$, the lattice points $\mathbf{x}, \mathbf{x}'$ satisfy

$$\sum_{i=1}^{n} x_i \cdot b_i = \sum_{i=1}^{n} x_i' \cdot b_i = 0$$

Geometry of Sidon Sets

- Hence, we must have

$$\sum_{i=1}^{n} f_i \cdot b_i = \sum_{i=1}^{n} f_i' \cdot b_i$$

for two different vectors $\mathbf{f}, \mathbf{f}'$ in the simplex $\triangle_h^n$

Geometry of Sidon Sets

- Hence, we must have

$$\sum_{i=1}^{n} f_i \cdot b_i = \sum_{i=1}^{n} f_i' \cdot b_i$$

for two different vectors $\mathbf{f}, \mathbf{f}'$ in the simplex $\triangle_h^n$

- Written differently

$$f_1 \cdot b_1 + \cdots + f_n \cdot b_n = f_1' \cdot b_1 + \cdots + f_n' \cdot b_n$$

where $f_i, f_i' \geq 0$, $\sum_{i=1}^{n} f_i \leq h$, $\sum_{i=1}^{n} f_i' \leq h$

Geometry of Sidon Sets

- Hence, we must have

$$\sum_{i=1}^{n} f_i \cdot b_i = \sum_{i=1}^{n} f_i' \cdot b_i$$

for two different vectors $\mathbf{f}, \mathbf{f}'$ in the simplex $\triangle_h^n$

- Written differently

$$f_1 \cdot b_1 + \cdots + f_n \cdot b_n = f_1' \cdot b_1 + \cdots + f_n' \cdot b_n$$

where $f_i, f_i' \geq 0$, $\sum_{i=1}^{n} f_i \leq h$, $\sum_{i=1}^{n} f_i' \leq h$

- This means that $\{0, b_1, \ldots, b_n\}$ is not a Sidon set of order h

Geometry of Sidon Sets

- Hence, we must have

$$\sum_{i=1}^{n} f_i \cdot b_i = \sum_{i=1}^{n} f_i' \cdot b_i$$

for two different vectors $\mathbf{f}, \mathbf{f}'$ in the simplex $\triangle_h^n$

- Written differently

$$f_1 \cdot b_1 + \cdots + f_n \cdot b_n = f_1' \cdot b_1 + \cdots + f_n' \cdot b_n$$

where $f_i, f_i' \geq 0$, $\sum_{i=1}^{n} f_i \leq h$, $\sum_{i=1}^{n} f_i' \leq h$

- This means that $\{0, b_1, \ldots, b_n\}$ is not a Sidon set of order h ■

Bounds

Bounds

- Let $\phi(h, n)$ denote the size of the smallest Abelian group containing a Sidon set of order h and cardinality $n + 1$

Bounds

- Let $\phi(h,n)$ denote the size of the smallest Abelian group containing a Sidon set of order h and cardinality $n+1$
- Let $K \subset \mathbb{R}^n$ be a compact convex set with non-empty interior. $(K,\mathcal{L})$ is a lattice packing in $\mathbb{R}^n$ if $K+\mathbf{x}$ and $K+\mathbf{y}$ have no interior points in common for all $\mathbf{x} \neq \mathbf{y} \in \mathbb{R}^n$. The lattice packing density is
$$\delta_{\mathrm{L}}(K) = \sup_{\mathcal{L}} \frac{\mathrm{Vol}(K)}{\det(\mathcal{L})}.$$

Bounds

- Let $\phi(h, n)$ denote the size of the smallest Abelian group containing a Sidon set of order h and cardinality $n+1$
- Let $K \subset \mathbb{R}^n$ be a compact convex set with non-empty interior. $(K, \mathcal{L})$ is a lattice packing in $\mathbb{R}^n$ if $K + \mathbf{x}$ and $K + \mathbf{y}$ have no interior points in common for all $\mathbf{x} \neq \mathbf{y} \in \mathbb{R}^n$. The lattice packing density is
$$\delta_{\mathrm{L}}(K) = \sup_{\mathcal{L}} \frac{\mathrm{Vol}(K)}{\det(\mathcal{L})}.$$
- Let $\delta_{\mathrm{L}}(\triangle^n)$ denote the lattice packing density of the simplex
$$\triangle^n = \left\{ \mathbf{y} \in \mathbb{R}^n \ : \ y_i \geq 0, \ \sum_{i=1}^n y_i \leq 1 \right\}.$$

Bounds

- Let $\phi(h,n)$ denote the size of the smallest Abelian group containing a Sidon set of order h and cardinality $n+1$
- Let $K \subset \mathbb{R}^n$ be a compact convex set with non-empty interior. $(K,\mathcal{L})$ is a lattice packing in $\mathbb{R}^n$ if $K+\mathbf{x}$ and $K+\mathbf{y}$ have no interior points in common for all $\mathbf{x} \neq \mathbf{y} \in \mathbb{R}^n$. The lattice packing density is

$$\delta_{\mathrm{L}}(K) = \sup_{\mathcal{L}} \frac{\mathrm{Vol}(K)}{\det(\mathcal{L})}.$$

- Let $\delta_{\mathrm{L}}(\triangle^n)$ denote the lattice packing density of the simplex

$$\triangle^n = \left\{ \mathbf{y} \in \mathbb{R}^n \;:\; y_i \geq 0,\ \sum_{i=1}^n y_i \leq 1 \right\}.$$

- The following are known:

$$\delta_{\mathrm{L}}(\triangle^1) = 1, \qquad \delta_{\mathrm{L}}(\triangle^2) = \frac{2}{3}, \qquad \delta_{\mathrm{L}}(\triangle^3) = \frac{18}{49}$$

Bounds

Theorem

For every $n \geq 1$ *and* $\epsilon > 0$,

$$\frac{1}{n!\ \delta_{\mathrm{L}}(\triangle^n)} h^n \leq \phi(h,n) < \frac{1+\epsilon}{n!\ \delta_{\mathrm{L}}(\triangle^n)} h^n,$$

the lower bound being valid for every $h \geq 1$, *and the upper bound for* $h \geq h_0(n,\epsilon)$.

Bounds

Theorem

For every $n \geq 1$ *and* $\epsilon > 0$,

$$\frac{1}{n!\ \delta_{\mathrm{L}}(\triangle^n)} h^n \leq \phi(h, n) < \frac{1+\epsilon}{n!\ \delta_{\mathrm{L}}(\triangle^n)} h^n,$$

the lower bound being valid for every $h \geq 1$*, and the upper bound for* $h \geq h_0(n, \epsilon)$*. Consequently,*

$$\lim_{h \to \infty} \frac{\phi(h, n)}{h^n} = \frac{1}{n!\ \delta_{\mathrm{L}}(\triangle^n)}.$$

Bounds

Theorem

For every $n \geq 1$ *and* $\epsilon > 0$,

$$\frac{1}{n!\ \delta_{\mathrm{L}}(\triangle^n)} h^n \leq \phi(h,n) < \frac{1+\epsilon}{n!\ \delta_{\mathrm{L}}(\triangle^n)} h^n,$$

the lower bound being valid for every $h \geq 1$, *and the upper bound for* $h \geq h_0(n,\epsilon)$. *Consequently,*

$$\lim_{h\to\infty} \frac{\phi(h,n)}{h^n} = \frac{1}{n!\ \delta_{\mathrm{L}}(\triangle^n)}.$$

Proof idea:

- Packing $\triangle_h^n$ in $\mathbb{Z}^n$ is equivalent to packing $\triangle_1^n$ in $\frac{1}{h}\mathbb{Z}^n$
- As $h \to \infty$, we get finer and finer grids $\frac{1}{h}\mathbb{Z}^n$ which approximate $\mathbb{R}^n$ ■

Bounds

- This gives the exact asymptotic (as $h \to \infty$) behavior of $\phi(h,n)$ for $n = 1, 2, 3$:

$$\phi(h,1) \sim h, \qquad \phi(h,2) \sim \frac{3}{4}h^2, \qquad \phi(h,3) \sim \frac{49}{108}h^3$$

and the best known bounds on $\phi(h,n)$ for $n \geq 4$.

Bounds

- This gives the exact asymptotic (as $h \to \infty$) behavior of $\phi(h,n)$ for $n = 1,2,3$:

$$\phi(h,1) \sim h, \qquad \phi(h,2) \sim \frac{3}{4}h^2, \qquad \phi(h,3) \sim \frac{49}{108}h^3$$

and the best known bounds on $\phi(h,n)$ for $n \geq 4$.

- In particular, for $n \to \infty$, we have

$$\lim_{h\to\infty} \frac{\phi(h,n)}{h^n} \leq \mathcal{O}((4e)^n n^{-n-2})$$

Significant improvement over Jia (J. Number Th., 1993)

$$\lim_{h\to\infty} \frac{\phi(h,n)}{h^n} \leq 1$$

Bases of order h

Bases of order h

Definition

Let G be a finite Abelian group. A subset $C = \{c_0, c_1, \ldots, c_n\} \subseteq G$ is said to be a basis of order h (or h-basis) of G if every element of the group can be expressed as $c_{i_1} + \ldots + c_{i_h}$ for some $0 \le i_1 \le \ldots \le i_h \le n$.

Bases of order h

Definition

Let G be a finite Abelian group. A subset $C = \{c_0, c_1, \dots, c_n\} \subseteq G$ is said to be a basis of order h (or h-basis) of G if every element of the group can be expressed as $c_{i_1} + \ldots + c_{i_h}$ for some $0 \le i_1 \le \ldots \le i_h \le n$.

Theorem

If $C = \{0, c_1, \dots, c_n\} \subseteq G$ is an h-basis for an Abelian group G, then $(\triangle_h^n, \mathcal{L})$ is a covering of $\mathbb{Z}^n$, where

$$\mathcal{L} = \left\{ \mathbf{x} \in \mathbb{Z}^n : \sum_{i=1}^{n} x_i c_i = 0 \right\},$$

and G is isomorphic to $\mathbb{Z}^n/\mathcal{L}$. Conversely, if $(\triangle_h^n, \mathcal{L}')$ is a lattice covering of $\mathbb{Z}^n$, then the group $\mathbb{Z}^n/\mathcal{L}'$ contains an h-basis of cardinality at most $n+1$.

Bounds on Bases of order h

Let $\psi(h, n)$ be the size of the largest Abelian group containing an h-basis of size $n + 1$.

Bounds on Bases of order h

Let $\psi(h,n)$ be the size of the largest Abelian group containing an h-basis of size $n+1$.

Theorem

For every fixed $n \geq 1$,

$$\lim_{h\to\infty} \frac{\psi(h,n)}{h^n} = \frac{1}{n!\vartheta_{\mathrm{L}}(\triangle^n)}$$

where $\vartheta_{\mathrm{L}}(\triangle^n)$ is the lattice covering density of $\triangle^n$.

Bounds on Bases of order h

Let $\psi(h,n)$ be the size of the largest Abelian group containing an h-basis of size $n+1$.

Theorem

For every fixed $n \geq 1$,

$$\lim_{h\to\infty} \frac{\psi(h,n)}{h^n} = \frac{1}{n!\vartheta_{\mathrm{L}}(\triangle^n)}$$

where $\vartheta_{\mathrm{L}}(\triangle^n)$ is the lattice covering density of $\triangle^n$.

$\vartheta_{\mathrm{L}}(\triangle^n)$ known for $n = 1, 2$:

$$\vartheta_{\mathrm{L}}(\triangle^1) = 1, \quad \text{and} \quad \vartheta_{\mathrm{L}}(\triangle^2) = \frac{3}{2}.$$

and for $n \geq 3$,

$$1 + 2^{-(3n+7)} \leq \vartheta_{\mathrm{L}}(\triangle^n) \leq n^{\log_2 \log_2 n + c}.$$

Conclusion

- Exact characterization of $\phi(h,n)$ which improves on existing bounds

$$\lim_{h\to\infty} \frac{\phi(h,n)}{h^n} = \frac{1}{n!\delta_{\mathrm{L}}(\triangle^n)}.$$

Conclusion

- Exact characterization of $\phi(h, n)$ which improves on existing bounds

$$\lim_{h \to \infty} \frac{\phi(h, n)}{h^n} = \frac{1}{n! \delta_{\mathrm{L}}(\triangle^n)}.$$

- Extensions to lattice coverings of simplices (Bases of order h)
  - "Improved Bounds on Sidon Sets via Lattice Packing of Simplices", M. Kovačević and V. Y. F. Tan, SIAM J. on Discrete Mathematics, Sep 2017

Conclusion

- Exact characterization of $\phi(h,n)$ which improves on existing bounds

$$\lim_{h\to\infty} \frac{\phi(h,n)}{h^n} = \frac{1}{n!\delta_{\mathrm{L}}(\triangle^n)}.$$

- Extensions to lattice coverings of simplices (Bases of order h)
  - "Improved Bounds on Sidon Sets via Lattice Packing of Simplices", M. Kovačević and V. Y. F. Tan, SIAM J. on Discrete Mathematics, Sep 2017

- Utility in coding for permutation channels (multiset codes) with deletions
  - "Codes in the Space of Multisets–Coding for Permutation Channels with Impairments", M. Kovačević and V. Y. F. Tan, IEEE Trans. on Inf. Th., to appear in 2018