On Binary Codes and Non-Interactive Simulation

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Non-Interactive Simulation Problem

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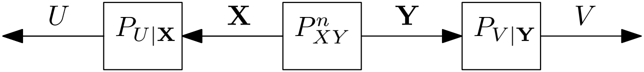

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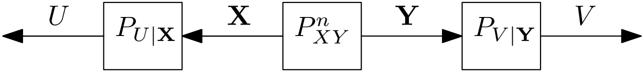

- A natural question: What are the possible joint distributions P_{UV} of (U, V)?

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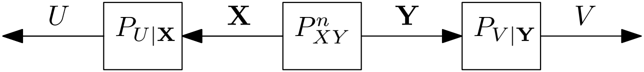

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- This problem is termed Non-Interactive Simulation of Random Variables

Background and Motivation

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- Used to define common information
  - Gács-Körner (1972) restricted U, V s.t. $\mathbb{P}(U = V) \to 1$ as $n \to \infty$
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- Noise-sensitivity of Boolean functions (Mossel-O'Donnell 2005):
  - $X \sim \mathrm{Bern}\left(\frac{1}{2}\right)$, $Y = X \oplus E$ with $E \sim \mathrm{Bern}(p)$ ind. of X
  - $U = f(\mathbf{X})$, $V = f(\mathbf{Y})$ with $f : \{-1, 1\}^n \to \{-1, 1\}$ being a balanced Boolean function (i.e., $\mathbb{P}(U = 1) = \mathbb{P}(V = 1) = \frac{1}{2}$)
  - maximize $\mathbb{P}(U = V)$ (or $\mathbb{E}UV$)

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- So in this work, we focus on the binary case:
  - X, Y, U, V are Boolean random variables taking values in $\{-1, 1\}$
  - P_{XY} is a Boolean symmetric distribution with correlation coefficient $\rho \in [0, 1]$, i.e.,

$$P_{XY} = \begin{array}{c} \\ -1 \\ 1 \end{array} \begin{array}{c} \begin{array}{cc} -1 & 1 \end{array} \\ \begin{bmatrix} \frac{1+\rho}{4} & \frac{1-\rho}{4} \\ \frac{1-\rho}{4} & \frac{1+\rho}{4} \end{bmatrix} \end{array}$$

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- If we restrict $U = f(\mathbf{X}), V = g(\mathbf{Y})$ for $f, g : \{-1,1\}^n \to \{-1,1\}$, we obtain

$$q_n^+(a,b) := \max_{\substack{f,g: \mathbb{P}(f(\mathbf{X})=1)=a_n, \\ \mathbb{P}(g(\mathbf{Y})=1)=b_n}} \mathbb{P}(f(\mathbf{X}) = g(\mathbf{Y}) = 1)$$

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where $a_n := \frac{\lfloor 2^n a \rfloor}{2^n}$ and $b_n := \frac{\lfloor 2^n b \rfloor}{2^n}$.

Replace $(P_{U|\mathbf{X}}, P_{V|\mathbf{Y}})$ with Boolean functions (f, g)

Lemma

We have

$$0 \le p_n^+(a,b) - q_n^+(a,b) \le 2^{-(n-1)}$$

$$0 \le p_n^-(a,b) - q_n^-(a,b) \le 2^{-(n-1)}.$$

In particular, if $a = \frac{M}{2^n}$ *and* $b = \frac{N}{2^n}$ *for some* $M, N \in \mathbb{N}$*, then*

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- Restricting $U = f(\mathbf{X}), V = g(\mathbf{Y})$ is asymptotically optimal in attaining $p_n^+(a,b), p_n^-(a,b)$

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  - In particular, if $A = B$, then

$$P^{(A,A)}(i) := \frac{1}{|A|^2} \left|\left\{\left(\mathbf{x}, \mathbf{x}'\right) \in A^2 : d_{\mathrm{H}}\left(\mathbf{x}, \mathbf{x}'\right) = i\right\}\right|, \quad i \in \{0, 1, ..., n\}$$

is the distance distribution of a single code $A \subseteq \{-1,1\}^n$

Distance Enumerators and Average Distances

- Define the distance enumerator between $A, B \subseteq \{-1, 1\}^n$ as

$$\Gamma_z(A, B) := \frac{1}{|A||B|} \sum_{\mathbf{x}\in A} \sum_{\mathbf{x}'\in B} z^{d_{\mathrm{H}}(\mathbf{x},\mathbf{x}')} = \sum_{i=0}^{n} P^{(A,B)}(i) \cdot z^i.$$

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$$D(A,B) := \frac{1}{|A||B|}\sum_{\mathbf{x}\in A}\sum_{\mathbf{x}'\in B} d_{\mathrm{H}}(\mathbf{x},\mathbf{x}') = \sum_{i=0}^{n} P^{(A,B)}(i)\cdot i$$

  - Clearly, $D(A,B)$ is the mean of $P^{(A,B)}$.

Equivalence

Lemma

For $a = \frac{M}{2^n}$ *and* $b = \frac{N}{2^n}$ *for some* $M, N \in \mathbb{N}$*, we have*

$$\mathbb{P}\left(f(\mathbf{X}) = g(\mathbf{Y}) = 1\right) = ab\,(1+\rho)^n\,\Gamma_{\frac{1-\rho}{1+\rho}}\left(A, B\right) = ab\Pi_\rho\left(A, B\right)$$

where $A := \{\mathbf{x} : f(\mathbf{x}) = 1\}$ *and* $B := \{\mathbf{x} : g(\mathbf{x}) = 1\}$*.*

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- Given a, b, ρ, characterizing the possible range of $\mathbb{P}(f(\mathbf{X}) = g(\mathbf{Y}) = 1)$ is equivalent to characterizing the possible range of $\Gamma_{\frac{1-\rho}{1+\rho}}(A, B)$ or $\Pi_\rho(A, B)$

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- The (Boolean function version of) non-interactive simulation problem $\Longleftrightarrow$ the problem of determining the possible range of the (dual) distance enumerator

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Theorem (Symmetric Case: $a = b$)

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where

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- Our bounds also hold for $q := \mathbb{P}(U = V = 1)$ (stochastic version).
- Our results for asymmetric cases can be found in our paper.

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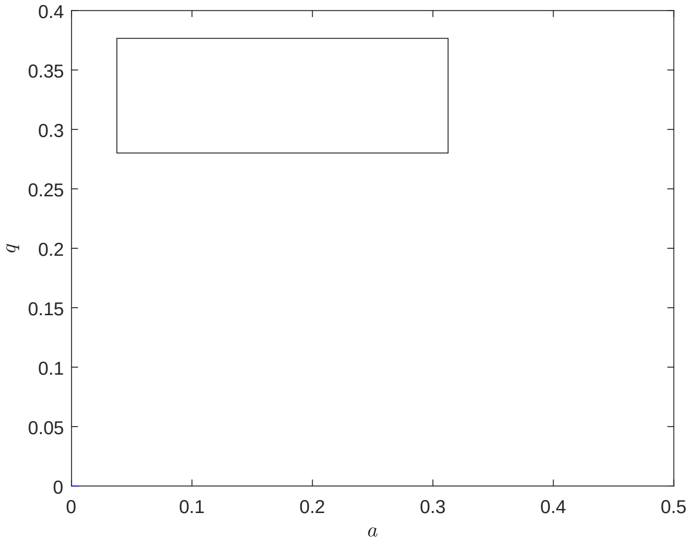

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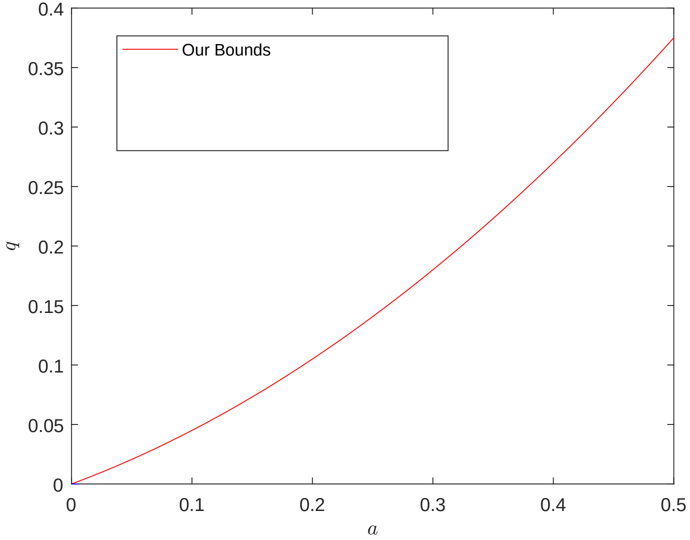

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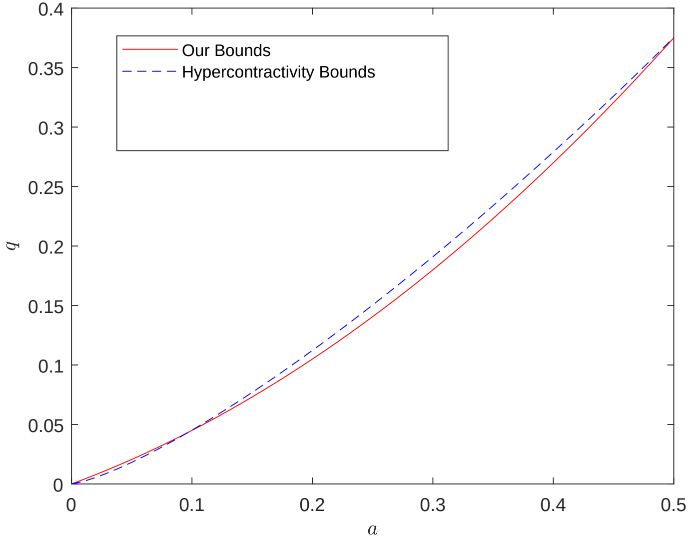

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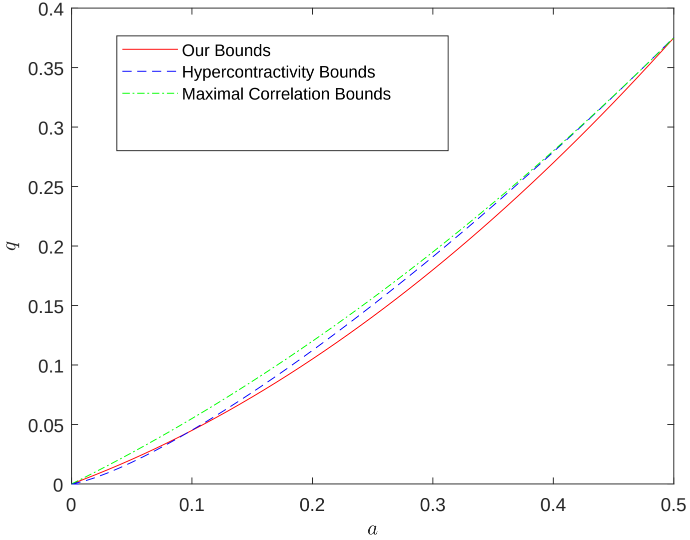

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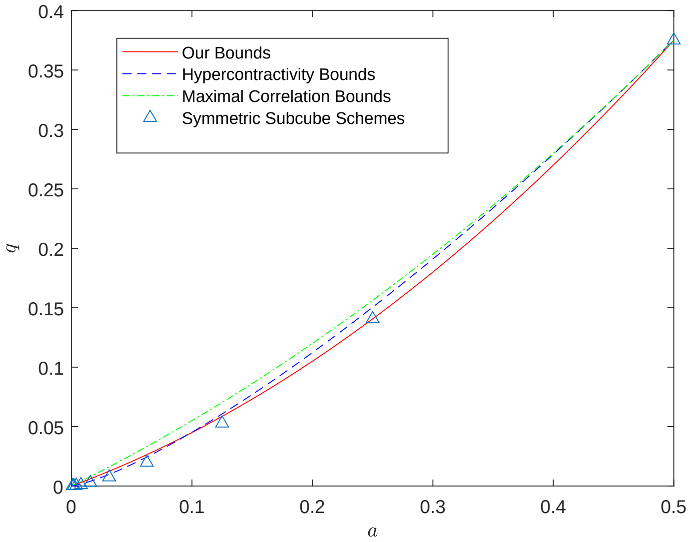

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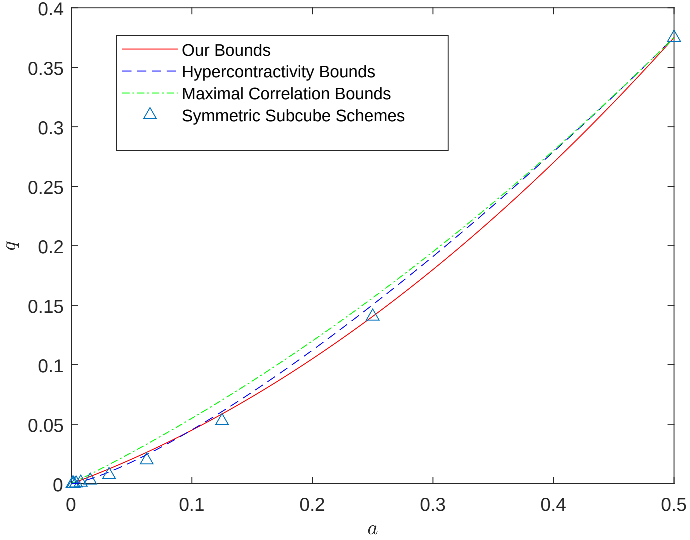

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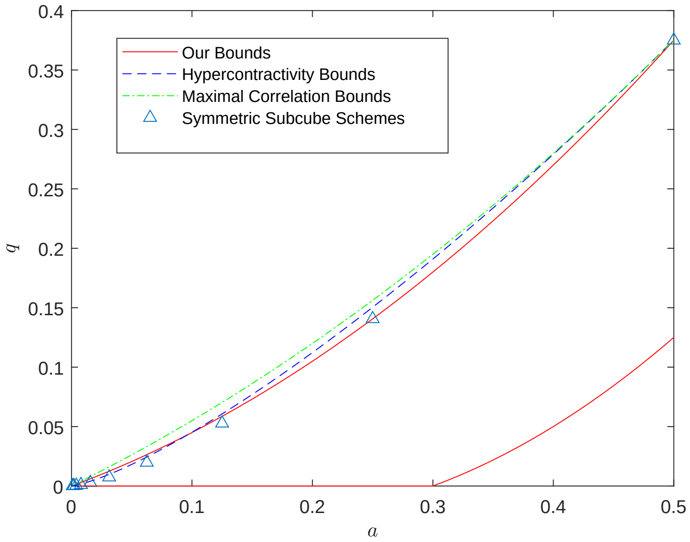

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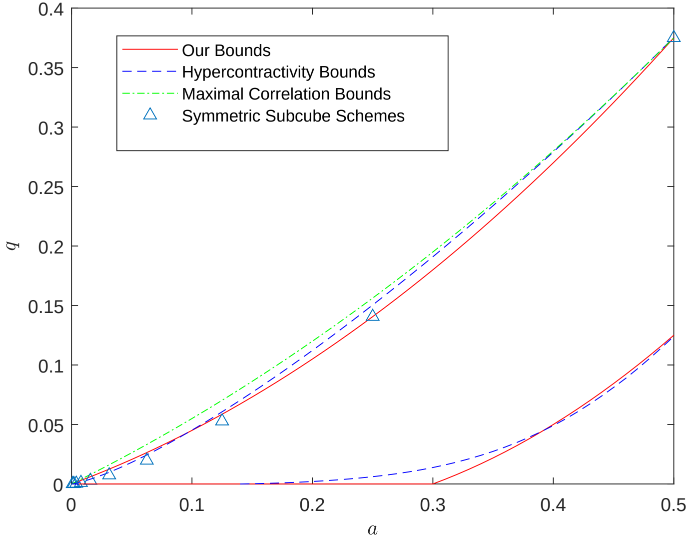

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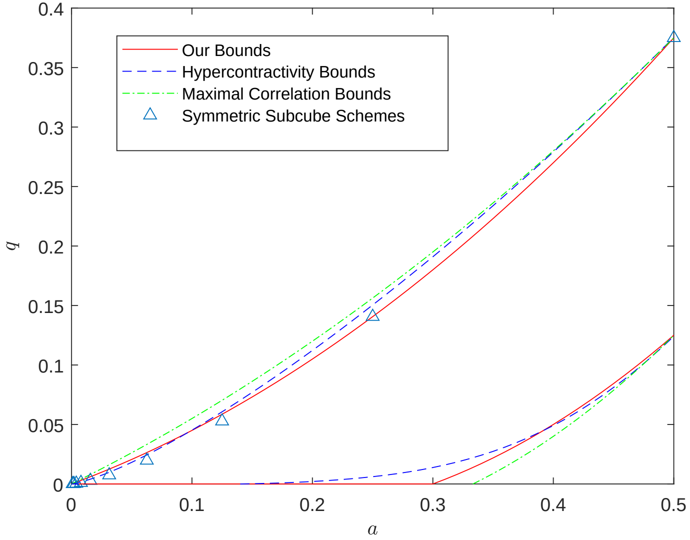

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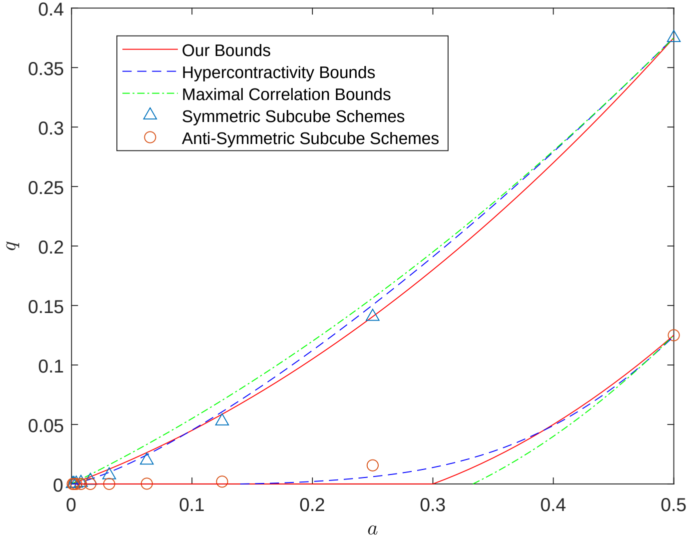

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$$\mathbb{P}(f(\mathbf{X}) = g(\mathbf{Y}) = 1) = ab + \frac{1}{4} \sum_{k=1}^{n} Q(k)\rho^k$$

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  - Combining the results above gives
$$|Q(1)| \le a + b$$

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- Finally, combining Steps 1 and 2 yields our bounds: $\theta^-(a) \leq q \leq \theta^+(a)$, where

$$\theta^+(a) = \min\left\{a, a^2 + \frac{a}{2}\rho + \left(\frac{a}{2} - a^2\right)\rho^2\right\}$$

$$\theta^-(a) = \max\left\{0, a^2 - \frac{a}{2}\rho - \left(\frac{a}{2} - a^2\right)\rho^2\right\}.$$

Why our proof works?

- In our Step 2, we use the following bounds:

$$\sum_{k=2}^{n} Q(k)\rho^k = \sum_{S\subseteq[n]:|S|\geq 2} \hat{f}_S \hat{g}_S \rho^{|S|} \in \left[\tau^- \rho^2, \tau^+ \rho^2\right]$$

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- Subcube functions satisfy $Q(k) = 0, k \geq m+1$

A New Bound on Average Distances

- Until now, we have shown the equivalence

$$\mathbb{P}(f(\mathbf{X}) = g(\mathbf{Y}) = 1) = ab(1+\rho)^n \Gamma_{\frac{1-\rho}{1+\rho}}(A, B) = ab\Pi_\rho(A, B)$$

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  - Non-interactive simulation is equivalent to some coding-theoretic problem
- We have applied coding-theoretic results to non-interactive simulation
- Next, in turn, we apply techniques for non-interactive simulation to a coding-theoretic problem
  - Specifically, apply hypercontractivity inequalities to bound average distances
- Recall that: The average distance between A, B is defined as

$$D(A,B) := \frac{1}{|A||B|}\sum_{\mathbf{x}\in A}\sum_{\mathbf{x}'\in B} d_{\mathrm{H}}(\mathbf{x},\mathbf{x}') = \sum_{i=0}^{n} P^{(A,B)}(i)\cdot i$$

Main Result: A New Bound on Average Distances

By hypercontractivity inequalities, we obtain:

Theorem

For $1 \le M \le 2^n$*, we have*

$$\min_{A:|A|=M} D(A,A) \ge \frac{n}{2} - \psi(a),$$

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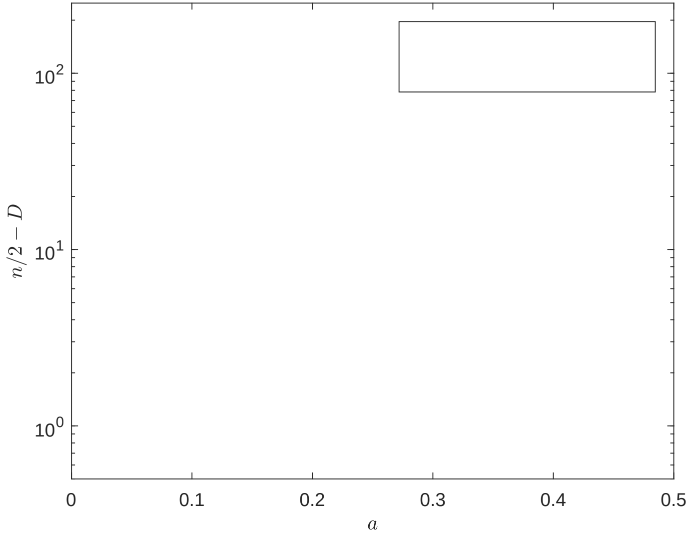

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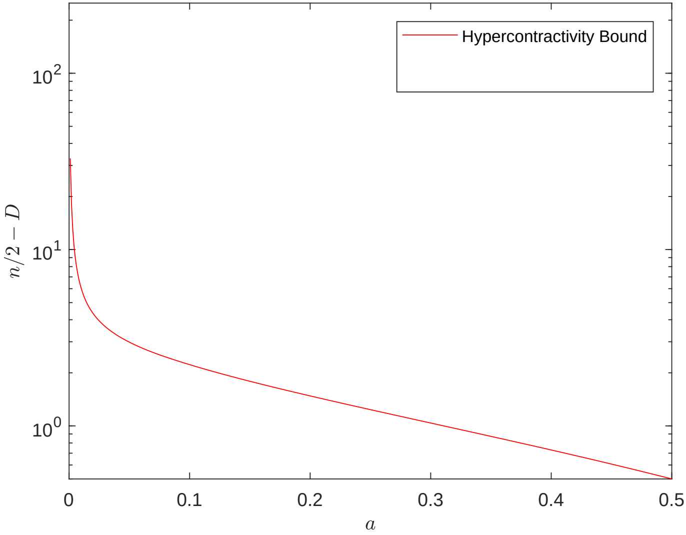

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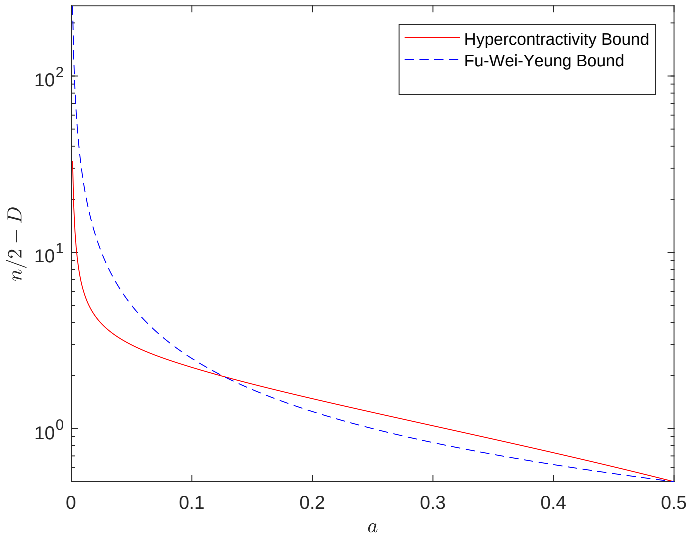

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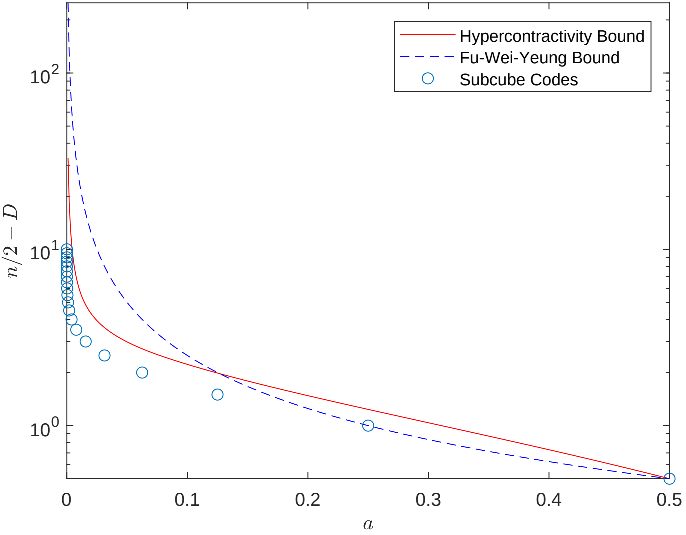

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- We applied Fourier analysis (combined with linear programming) to the non-interactive simulation problem
  - Our bounds are sharp for some cases and tighter than existing results for some other cases
- In turn, applied DPIs (hypercontractivity) to the minimal average-distance problem
  - Our bound is tighter than the best known result for some cases