Two Applications of the Gaussian Poincaré Inequality in the Shannon Theory

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***Abstract*—We employ the Gaussian Poincaré inequality for two tasks in the Shannon theory. First, we show that the Gaussian broadcast channel admits a strong converse. Second, we demonstrate that the empirical output distribution of a delay-limited code for the AWGN channel with quasi-static fading and with non-vanishing probability of error converges to the maximum mutual information output distribution (in the normalized relative entropy sense).**

I. Introduction

The Poincaré inequality for Gaussian measures [1] is one of the most prominent results in the theory of concentration of measure. Roughly speaking, it states that if $f : \mathbb{R}^n \to \mathbb{R}$ is a smooth function and $\phi(x) = \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-x^2/2}$ is the standard Gaussian density, then the variance of f can be bounded in terms of the expectation of the squared derivative of f, i.e.,

$$\mathsf{Var}_\phi[f] \le \mathbb{E}_\phi[\|\nabla f\|^2]. \tag{1}$$

In the present work, we employ a modification of the Gaussian Poincaré inequality for two tasks in Shannon theory. These are described briefly in the following sections.

II. Gaussian Broadcast Channels

The Gaussian broadcast channel [2, Ch. 5] is a basic model for the downlink of a communication system. Two messages $W_1 \in [2^{nR_1}]$ and $W_2 \in [2^{nR_2}]$ are to be encoded into a codeword $X^n = f^{(n)}(W_1, W_2)$. This codeword is power constrained, i.e., $\|X^n\|_2^2 \le nP$. It is transmitted through two AWGN channels with variances σ_1^2 and σ_2^2 respectively, i.e.,

$$Y_1^n = X^n + Z_1^n, \quad \text{and} \quad Y_2^n = X^n + Z_2^n. \tag{2}$$

Decoder j, which observes Y_j^n, is required to estimate message W_j where $j = 1, 2$. The average probability of error is defined to be $\Pr((\hat{W}_1, \hat{W}_2) \ne (W_1, W_2))$ where $\hat{W}_j$ is decoder j's estimate of W_j. The capacity region $\mathcal{C}_{\mathrm{BC}}$ is well known and is given by

$$\mathcal{C}_{\mathrm{BC}} = \bigcup_{\alpha \in [0,1]} \left\{ (R_1, R_2) \in \mathbb{R}_+^2 \,\middle|\, \begin{array}{l} R_1 \le \mathrm{C}\left(\frac{\alpha P}{\sigma_1^2}\right) \\ R_2 \le \mathrm{C}\left(\frac{(1-\alpha)P}{\alpha P + \sigma_2^2}\right) \end{array} \right\}, \tag{3}$$

where $\mathrm{C}(x) := \frac{1}{2}\log(1+x)$. This region is achieved using superposition coding [3]. Recall that the capacity region is the set of all rate pairs for which the error probability vanishes.

The central question of our investigation in [4] is whether the region in (3) is enlarged if we relax the condition that the error probability vanishes. We allow the error probability to be upper bounded by a non-vanishing constant $\varepsilon \in (0, 1)$. We show that the ε-capacity region is precisely the region in (3). The main technicality in the proof involves bounding a certain variance of the log-likelihood of the messages using (1).

III. Good Delay-Limited Codes

In [5], we used (1) to investigate quasi-static fading channels [6, Sec. 5.4.1] where the fading coefficient H is random but remains constant during the course of transmission. We are interested in the so-called *delay-limited capacity* [7], which is the maximum achievable rate under the assumption that the maximal error probability over all non-zero fading coefficients vanishes as the blocklength grows.

We adopt a long-term power constraint [8] and the max-over-messages error criterion for delay-limited decoding. It is known (e.g., [7, Sec. III-B]) that the delay-limited capacity is $\mathrm{C}(P_{\mathrm{DL}})$ where $P_{\mathrm{DL}} := \frac{P}{\mathbb{E}[1/H]}$. We show in [5] that for any sequence of codes that is capacity-achieving and whose error probability is upper bounded by some $\varepsilon \in [0, 1)$ is such that sequence of induced output distributions $\{p_{Y^n}\}_{n=1}^\infty$ satisfies

$$\lim_{n\to\infty} \frac{1}{n} D(p_{Y^n} \| p_{Y^*}^n) = 0 \tag{4}$$

where $p_{Y^*}(y) = \mathcal{N}(y; 0, 1 + P_{\mathrm{DL}})$.

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