The Dispersion of Nearest-Neighbor Decoding for Additive Non-Gaussian Channels

Jonathan Scarlett Laboratory for Information and Inference Systems, École Polytechnique Fédérale de Lausanne, Natio Email: jmscarlett@gmail.com

Vincent Y. F. Tan Department of ECE, National University of Singapore Email: vtan@nus.edu.sg Giuseppe Durisi Department of Signals and Systems, Chalmers University of Technology Email: durisi@chalmers.se

Abstract—We study the second-order asymptotics of information transmission using random Gaussian codebooks and nearest neighbor (NN) decoding over a power-limited additive stationary memoryless non-Gaussian channel. We show that the dispersion term depends on the non-Gaussian noise only through its second and fourth moments. We also characterize the second-order performance of point-to-point codes over Gaussian interference networks. Specifically, we assume that each user's codebook is Gaussian and that NN decoding is employed, i.e., that interference from unintended users is treated as noise at each decoder.

I. SYSTEM MODEL

Consider the point-to-point additive-noise channel

$$Y^n = X^n + Z^n,\tag{1}$$

where X^n is the input and Z^n is the noise over *n* scalar channel uses. Throughout, we shall focus exclusively on Gaussian codebooks. More precisely, we consider *shell codes* for which X^n is uniformly distributed on the sphere

$$X^n \sim \mathbf{f}_{X^n}^{(\text{shell})}(\mathbf{x}) := \delta(\|\mathbf{x}\|^2 - nP) / S_n(\sqrt{nP}).$$
(2)

Here, $\delta(\cdot)$ is the Dirac delta and $S_n(r) = 2\pi^{n/2}r^{n-1}/\Gamma(n/2)$ is the surface area of a radius-*r* sphere in \mathbb{R}^n . The noise Z^n is assumed to be a stationary and memoryless process that does not depend on the channel input: $Z^n \sim P_{Z^n}(\mathbf{z}) = \prod_{i=1}^n P_Z(z_i)$. The distribution P_Z is non-Gaussian; the only assumptions are:

$$\mathbb{E}[Z^2] = 1, \qquad \xi := \mathbb{E}[Z^4] < \infty, \qquad \mathbb{E}[Z^6] < \infty.$$
(3)

Given a shell code consisting of $M \in \mathbb{N}$ random codewords $\mathcal{C} := \{X^n(1), \ldots, X^n(M)\}$, we consider an nearest neighbor decoder that returns the message \hat{W} whose corresponding codeword is closest in Euclidean distance to Y^n , i.e.,

$$\hat{W} := \underset{w \in [1:M]}{\arg\min} \|Y^n - X^n(w)\|.$$
(4)

This decoder is optimal if the noise is Gaussian, but may not be so in the more general setup considered here.

We define the average probability of error as $\bar{p}_{e,n} := \Pr[\hat{W} \neq W]$. This probability is averaged over the uniformly distributed message W, the random codebook C and the channel noise Z^n . Note that in traditional channel-coding analyses [1], [2], the probability of error is averaged only over W and Z^n . Similar to [3], the additional averaging over the

codebook C is required here to establish an ensemble converse for the class of Gaussian codebooks considered in this paper.

Let $M_{\text{shell}}^*(n, \varepsilon, P; P_Z)$ be the maximum number of messages that can be transmitted using a shell codebook over the channel (1) with average error probability no larger than $\varepsilon \in (0, 1)$, when the noise is distributed according to P_Z Lapidoth [3] showed that for all $\varepsilon \in (0, 1)$,

$$\lim_{n \to \infty} \frac{1}{n} \log M_{\text{shell}}^*(n, \varepsilon, P; \mathbf{P}_Z) = \mathsf{C}(P).$$
 (5)

independent of P_Z .

In Theorem 1 below, we provide the second-order term in the asymptotic expansion of $\log M^*_{\text{shell}}(n, \varepsilon, P; \mathbf{P}_Z)$.

Theorem 1. Consider a noise distribution with statistics as in (3). For shell codes,

$$\log M_{\text{shell}}^*(n,\varepsilon,P;\mathbf{P}_Z) = n\mathsf{C}(P) - \sqrt{n\mathsf{V}_{\text{shell}}(P,\xi)}\mathbf{Q}^{-1}(\varepsilon) + O(\log n), \quad (6)$$

where the shell dispersion is

$$\mathsf{V}_{\rm shell}(P,\xi) := \left(P^2(\xi - 1) + 4P \right) / \left(4(P+1)^2 \right). \tag{7}$$

The proof together with an extension to Gaussian interference networks can be found in [4]. One of the main tools in our second-order analysis is the *Berry-Esseen theorem* for functions of random vectors (see, e.g., [5, Prop. 1]). The second-order term in the asymptotic expansions of $\log M_{\text{shell}}^*(n, \varepsilon, P; P_Z)$ depends on the distribution P_Z only through its second and fourth moments. If Z is standard Gaussian, then the fourth moment $\xi = 3$ and we recover from (7) the Gaussian dispersion [2, Eq. (293)]. Comparing (7) with [2, Eq. (293)] we see that noise distributions P_Z with higher fourth moments than Gaussian (e.g., Laplace) result in a slower convergence to C(P).

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