On Gaussian MACs with Feedback

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(Joint work with Lan V. Truong and Silas L. Fong)

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Tokyo University of Agriculture and Technology, Sept 2017

Outline

1. Background

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2. Fixed-Length Feedback for the AWGN Channel

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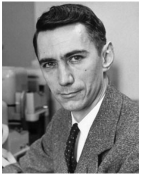
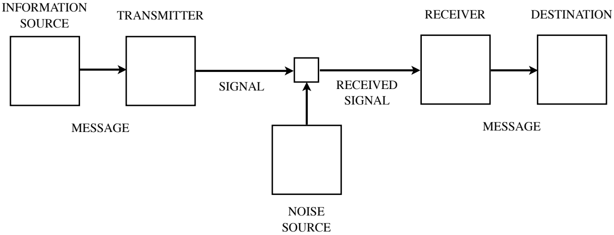
4. Variable-Length Feedback for the G-MAC

5. Conclusion

Information Transmission

- Shannon's information theory:





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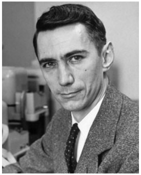

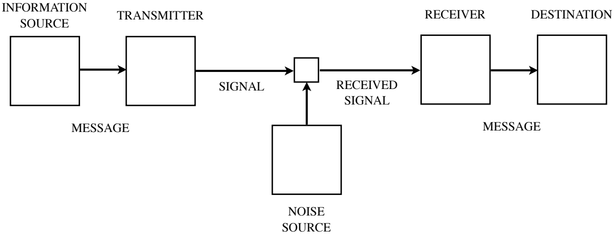

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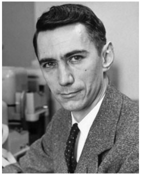

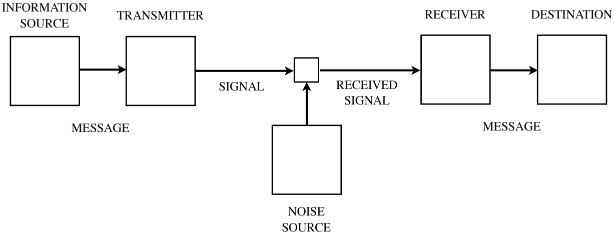

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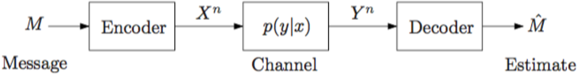
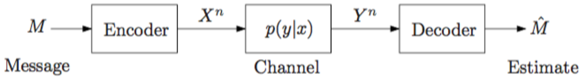
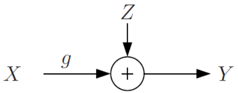
- "Feedback doesn't increase capacity" *[Shannon (1956)]*

AWGN Channel




M
Message
Encoder
X^n
$p(y|x)$
Channel
Y^n
Decoder
$\hat{M}$
Estimate


Z
X
g
+
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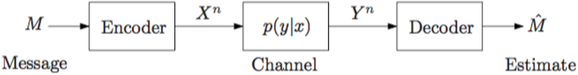

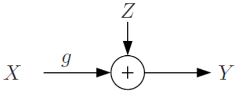

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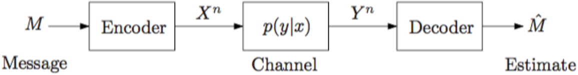

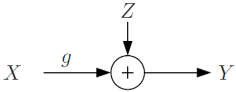

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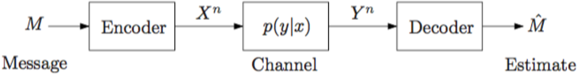

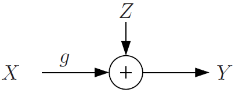

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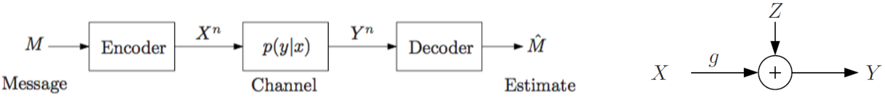

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- Expected or Long-Term power constraint

$$\frac{1}{\mathsf{M}}\sum_{m=1}^{\mathsf{M}}\left(\frac{1}{n}\sum_{i=1}^{n} X_i^2(m)\right) \le P.$$

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$$\lim_{\varepsilon\downarrow 0}\ \varliminf_{n\to\infty}\ \frac{1}{n}\log \mathsf{M}^*_{\mathrm{PP}}(n,P,\varepsilon) = \mathsf{C}(P),$$
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- In n channel uses, can send up to $n\mathsf{C}(P)$ nats over $p(y|x)$ reliably.

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- If we do not demand that the avg error prob. vanishes *[Yoshihara (1964), Polyanskiy-Poor-Verdú (2010)]*,

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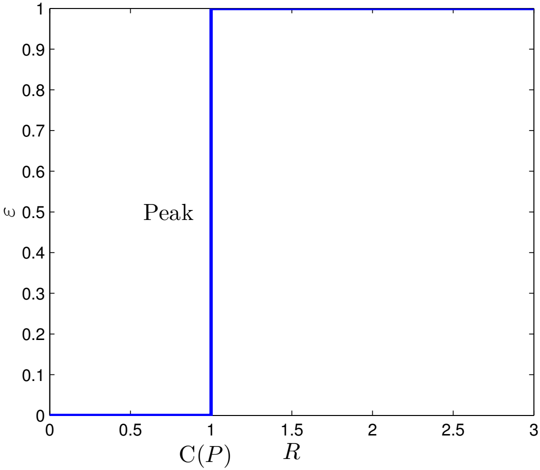
- Since for long-term, the ε-capacity depends on ε, the strong converse does not hold

Strong Converse?

$$\varepsilon = \varlimsup_{n\to\infty} \mathrm{P}_{\mathrm{e}}^{(n)}, \qquad R = \varliminf_{n\to\infty} \frac{1}{n} \log \mathsf{M}$$



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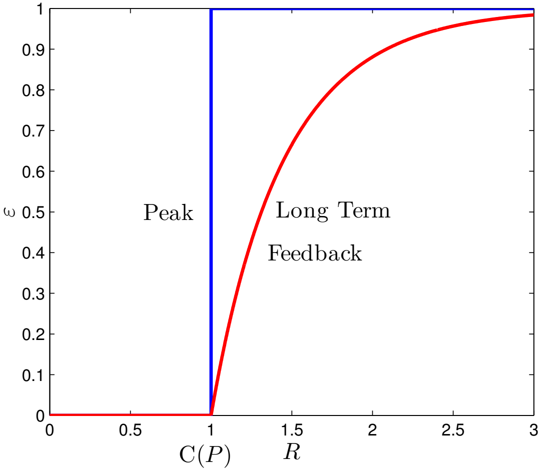

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$$\log \mathsf{M}^*_{\mathrm{PP}}(n, P, \varepsilon) = n\mathsf{C}(P) + \sqrt{n\mathsf{V}(P)}\Phi^{-1}(\varepsilon) + \frac{1}{2}\log n + O(1)$$

where the channel dispersion is

$$\mathsf{V}(x) := \frac{x(x+2)}{2(x+1)^2} \quad \text{squared nats per ch. use}$$

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$$\Phi(a) := \int_{-\infty}^{a} \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-t^2/2}\, \mathrm{d}t.$$

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- Second-order *[Yang-Caire-Durisi-Polyanskiy (2015)]*

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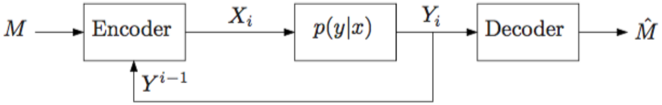
Feedback



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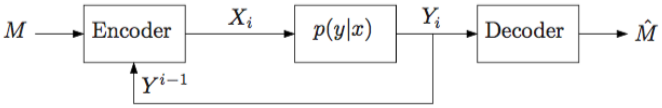

- Feedback helps to simplify coding schemes
- Long-term power constraint under feedback

$$\frac{1}{\mathsf{M}}\sum_{m=1}^{\mathsf{M}}\left(\frac{1}{n}\sum_{i=1}^{n}\mathbb{E}\left[X_i^2(m,Y^{i-1})\right]\right)\le P.$$

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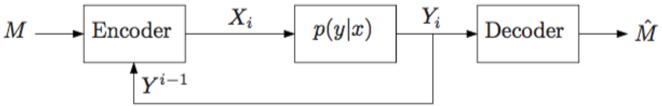

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Feedback : Existing Results

- First-order *[Shannon (1956)]*

$$\lim_{\varepsilon \downarrow 0} \varliminf_{n \to \infty} \frac{1}{n} \log \mathsf{M}^*_{\mathrm{FB}}(n, P, \varepsilon) = \mathsf{C}(P).$$

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- *Schalkwijk and Kailath (1966)* demonstrated a simple coding scheme based on estimation-theoretic ideas to show that

$$\mathrm{P}_{\mathrm{e}}^{(n)}(R) \le 2 \exp\left(-\frac{2^{2n(\mathsf{C}(P)-R)}}{2} \right), \quad \text{for} \quad R = \frac{1}{n} \log \mathsf{M} < \mathsf{C}(P).$$

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- Error exponent is infinity
- Suggests that the fixed-error results can also be drastically improved

AWGN Channels with Feedback : New Results

Theorem (Truong-Fong-T. (T-IT March 2017))

For the directpart,

$$\log \mathsf{M}^*_{\mathrm{FB}}(n, P, \varepsilon) \geq n\mathsf{C}\left(\frac{P}{1-\varepsilon}\right) - \log\log n + O(1).$$

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- From these results, the ε-capacity is

$$\lim_{n\to\infty}\frac{1}{n}\log \mathsf{M}^*_{\mathrm{FB}}(n,P,\varepsilon) = \mathsf{C}\Big(\frac{P}{1-\varepsilon}\Big).$$

- The $-\log\log n$ can be replaced by $-O(\underbrace{\log\ldots\log}_{L\text{ times}} n)$ for any $L \in \mathbb{N}$.

AWGN Channels with Feedback : Remarks

$$\lim_{n\to\infty} \frac{1}{n} \log \mathsf{M}^*_{\mathrm{FB}}(n, P, \varepsilon) = \mathsf{C}\Big(\frac{P}{1-\varepsilon}\Big).$$

- Feedback doesn't improve the first-order term since

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- With feedback, second-order term is at least

$$-O(\log\log\ldots\log n).$$

This is a great improvement over without feedback where the second-order term is *[Yang-Caire-Durisi-Polyanskiy (2015)]*

$$-\sqrt{\mathsf{V}\Big(\frac{P}{1-\varepsilon}\Big)}\sqrt{n\log n} + o(\sqrt{n}).$$

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- Choose

$$\log \mathsf{M}' = n\mathsf{C}\left(\frac{P}{1-\varepsilon}\right) - \log\log n + O_\varepsilon(1)$$

where $-\log\log n$ because of double exponential decay of $\mathrm{P}_{\mathrm{e}}^{(n)}(R)$

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where $-\log\log n$ because of double exponential decay of $\mathrm{P}_\mathrm{e}^{(n)}(R)$

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Proof Idea for the Direct Part

- Partition msg set $\{1, \ldots, \mathsf{M}\}$ into $\mathcal{A}_1 \sqcup \mathcal{A}_2$.
- $\mathcal{A}_1$: Send $(0, 0, \ldots, 0) \in \mathbb{R}^n$
- $\mathcal{A}_2$: *Schalkwijk-Kailath (1966)* scheme $\mathsf{M}' = |\mathcal{A}_2| \approx (1-\varepsilon)\mathsf{M}$ msg

$$\mathrm{P}_{\mathrm{e}}^{(n)}(R_n' \,|\, \mathcal{A}_2) \le n^{-1}, \quad \text{where} \quad R_n' := n^{-1} \log \mathsf{M}'.$$

- Choose

$$\log \mathsf{M}' = n\mathsf{C}\left(\frac{P}{1-\varepsilon}\right) - \log\log n + O_\varepsilon(1)$$

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- By *Gallager-Nakiboglŭ (2010)* and *Ihara (2012)*, the $-O(\log \ldots \log n)$ term arises because $\mathrm{P}_{\mathrm{e}}^{(n)}(R)$ decays faster than any multiple-exponential.
- Hence,

$$\mathrm{P}_{\mathrm{e}}^{(n)} = \Pr(\mathcal{A}_1)\mathrm{P}_{\mathrm{e}}^{(n)}(\mathcal{A}_1) + \Pr(\mathcal{A}_2)\mathrm{P}_{\mathrm{e}}^{(n)}(\mathcal{A}_2) \le \varepsilon \cdot 1 + (1-\varepsilon)\frac{1}{n} \approx \varepsilon.$$

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$\exists$ LT-FB code $\{X_i(\cdot,\cdot)\}_{i=1}^n$ with M msges and $\mathrm{P}_{\mathrm{e}}^{(n)} \le \varepsilon$

$\Longrightarrow \exists$ PP-FB code $\{X_i'(\cdot,\cdot)\}_{i=1}^n$ with M msges and $\mathrm{P}_{\mathrm{e}}^{(n)} \le 1 - \frac{1}{\sqrt{n}}$

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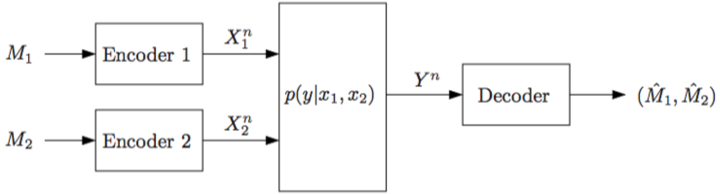
- Exploit connection between binary hypothesis testing and channel coding with feedback under peak-power constraint *[Polyanskiy-Poor-Verdú (2011)] [Fong-T. (2015)]*

Outline

MACs and Gaussian MACs

- The multiple access channel (MAC)



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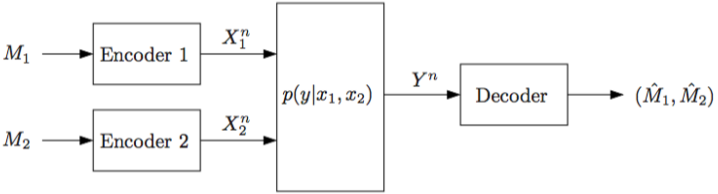
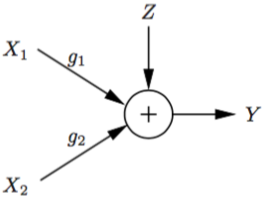
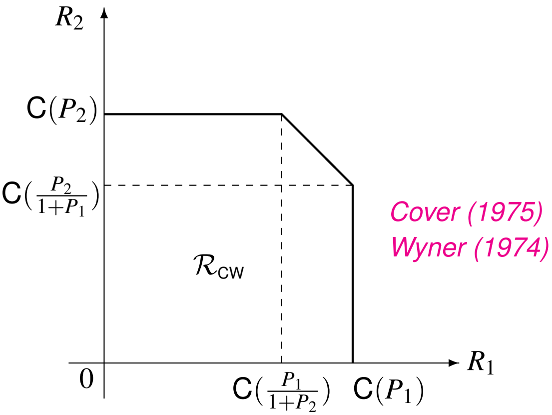


- The Gaussian multiple access channel



Again assume $g_1 = g_2 = 1$.

Capacity Region for the Gaussian MAC



$$R_1 \le \mathsf{C}(P_1)$$
$$R_2 \le \mathsf{C}(P_2)$$
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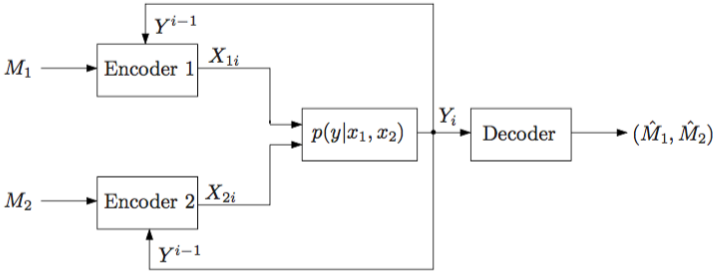
Gaussian MAC with Feedback



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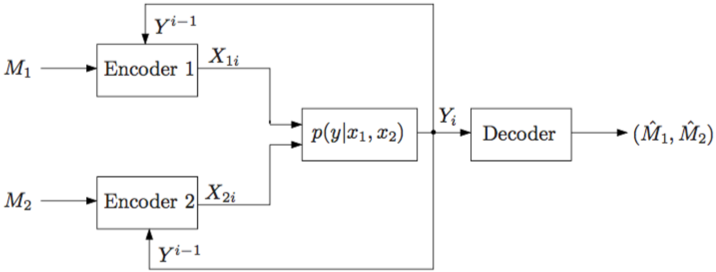

Consider Gaussian version with expected long-term power constraints

$$\frac{1}{n}\sum_{i=1}^{n}\mathbb{E}\left[X_{1i}^2(M_1, Y^{i-1})\right] \le P_1, \quad \frac{1}{n}\sum_{i=1}^{n}\mathbb{E}\left[X_{2i}^2(M_2, Y^{i-1})\right] \le P_2.$$

Capacity Region of the G-MAC with Feedback

- *Ozarow (1984)* showed that the capacity region is

$$\mathcal{R}_{\text{Ozarow}}(P_1, P_2) := \bigcup_{0 \le \rho \le 1} \left\{ (R_1, R_2) \;\middle|\; \begin{array}{l} R_1 \le \mathsf{C}\big((1-\rho^2)P_1\big), \\ R_2 \le \mathsf{C}\big((1-\rho^2)P_2\big), \\ R_1 + R_2 \le \mathsf{C}\Big(P_1 + P_2 + 2\rho\sqrt{P_1 P_2}\Big) \end{array} \right\}.$$

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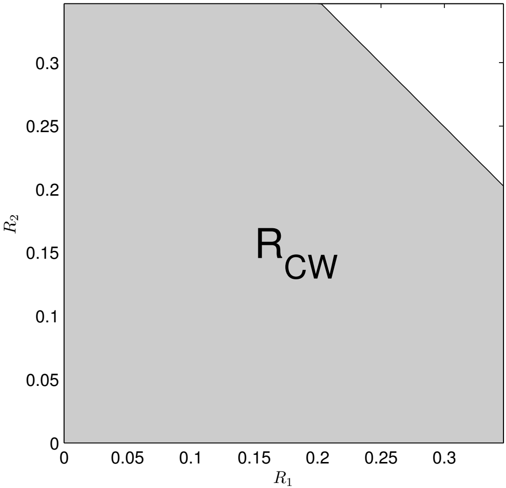
- Direct part is an extension of the Schalkwijk and Kailath coding scheme

CR of the G-MAC with Feedback $P_1 = P_2 = 1$



No feedback

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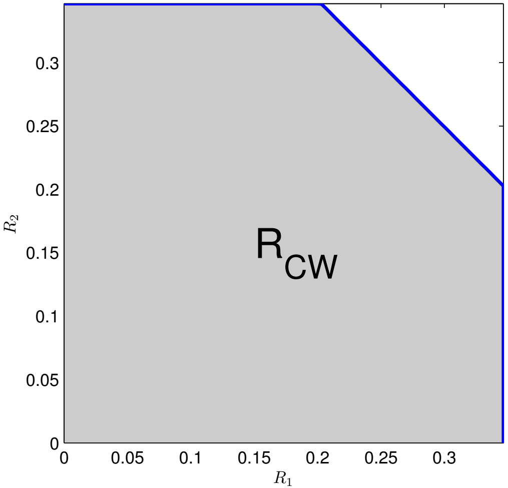

$$\rho = 0$$

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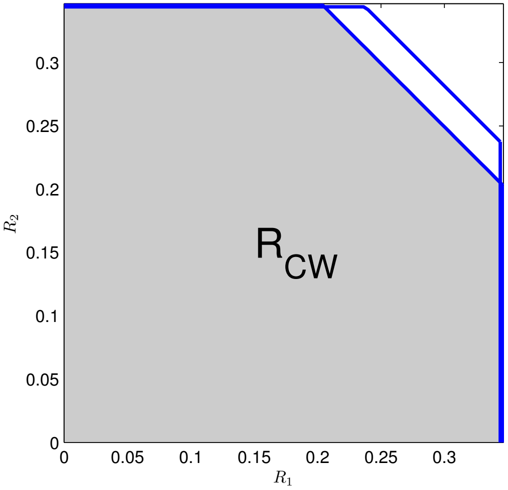

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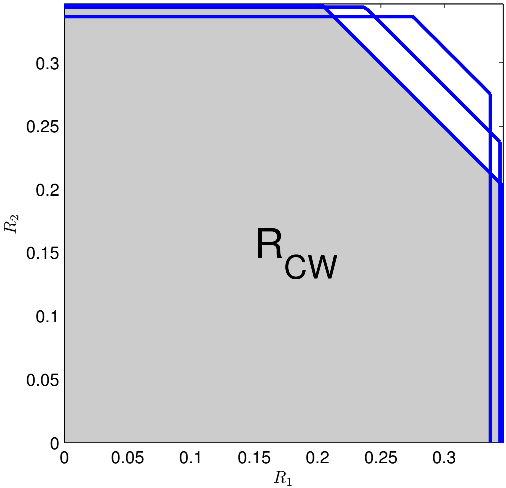

$$\rho = 0.2$$

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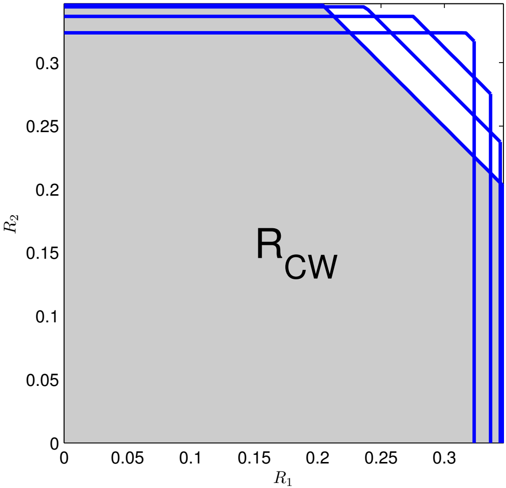
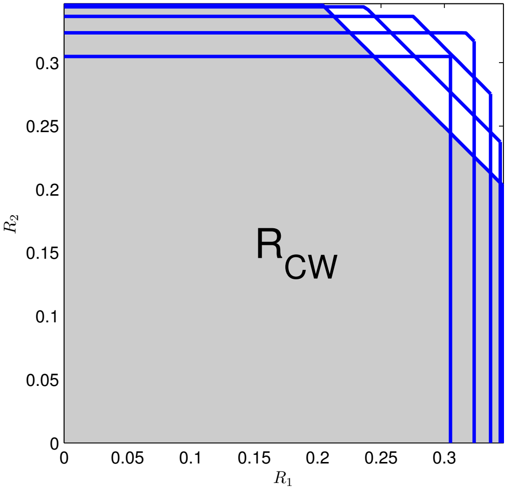


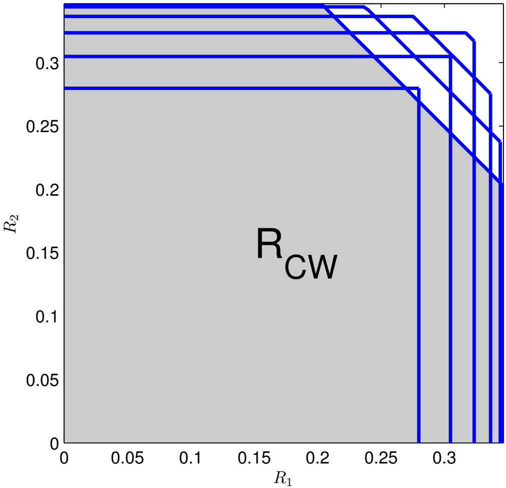
$$\rho = 0.3$$

CR of the G-MAC with Feedback $P_1 = P_2 = 1$



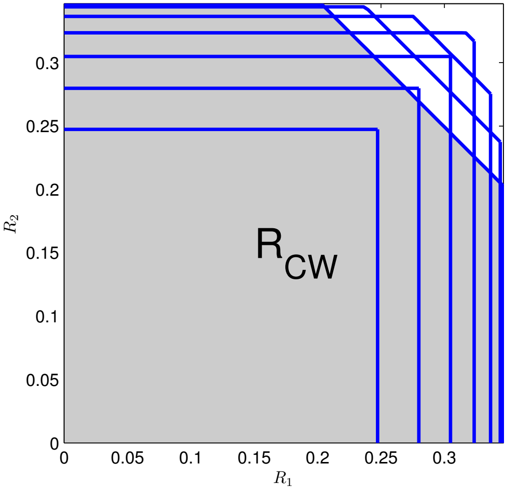
$$\rho = 0.4$$

CR of the G-MAC with Feedback $P_1 = P_2 = 1$



$$\rho = 0.5$$

CR of the G-MAC with Feedback $P_1 = P_2 = 1$



$$\rho = 0.6$$

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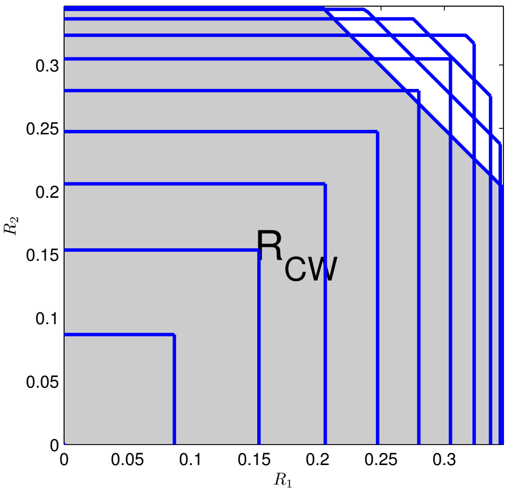
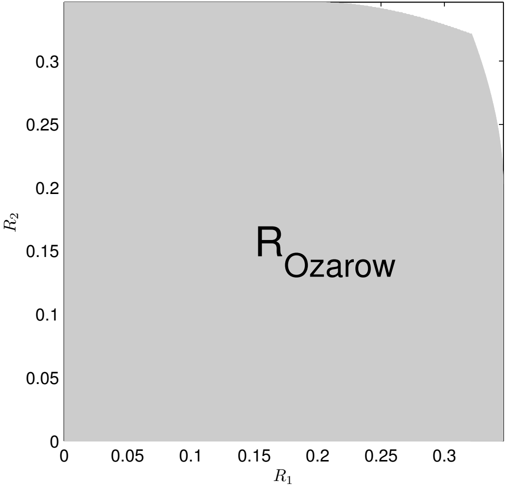


$\rho = 1.0$

CR of the G-MAC with Feedback $P_1 = P_2 = 1$



The Ozarow region

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$$\varliminf_{n\to\infty} \frac{1}{n} \log \mathsf{M}_1 \ge R_1 \qquad \varliminf_{n\to\infty} \frac{1}{n} \log \mathsf{M}_2 \ge R_2,$$

  and the average probability of error

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- $\mathcal{C}_\varepsilon(P_1, P_2)$ is the set of all ε-achievable (R_1, R_2).

ε-Capacity Region of the G-MAC with Feedback

Theorem (Truong-Fong-T. (T-IT March 2017))

The ε-capacity region is

$$\mathcal{C}_\varepsilon(P_1, P_2) = \mathcal{R}_{\mathrm{Ozarow}}\Big(\frac{P_1}{1-\varepsilon}, \frac{P_2}{1-\varepsilon}\Big), \quad \textit{for all} \quad \varepsilon \in [0, 1).$$

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If we can tolerate an error of $\le \varepsilon$, we can operate at (R_1, R_2) satisfying

$$\begin{aligned} R_1 &\le \mathsf{C}\left(\frac{(1-\rho^2)P_1}{1-\varepsilon}\right) \\ R_2 &\le \mathsf{C}\left(\frac{(1-\rho^2)P_2}{1-\varepsilon}\right), \qquad \text{for any } 0 \le \rho \le 1. \\ R_1 + R_2 &\le \mathsf{C}\left(\frac{P_1 + P_2 + 2\rho\sqrt{P_1 P_2}}{1-\varepsilon}\right) \end{aligned}$$

This is optimal.

ε-Capacity of the G-MAC with Feedback : Remarks

- $\varepsilon = 0$ recovers Ozarow's result

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$$\log \mathsf{M}_1 \le \gamma_1 - \log^+ \left[1 - \varepsilon - \Pr\left(\sum_{i=1}^n \log \frac{W(Y_i|X_{1i}, X_{2i})}{Q_{Y_i|X_{2i}}(Y_i|X_{2i})} \ge \gamma_1 \right) \right]$$

$$\log \mathsf{M}_2 \le \gamma_2 - \log^+ \left[1 - \varepsilon - \Pr\left(\sum_{i=1}^n \log \frac{W(Y_i|X_{1i}, X_{2i})}{Q_{Y_i|X_{1i}}(Y_i|X_{1i})} \ge \gamma_2 \right) \right]$$

$$\log(\mathsf{M}_1\mathsf{M}_2) \le \gamma_3 - \log^+ \left[1 - \varepsilon - \Pr\left(\sum_{i=1}^n \log \frac{W(Y_i|X_{1i}, X_{2i})}{Q_{Y_i}(Y_i)} \ge \gamma_3 \right) \right]$$

Given a code generating symbols $\{(X_{1i}(\mathsf{M}_1, Y^{i-1}), X_{2i}(\mathsf{M}_2, Y^{i-1}))\}_{i=1}^n$, let

Proof Idea for the Converse Part : Step 2

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$$P_{1i} := \mathbb{E}[X_{1i}^2], \quad P_{2i} := \mathbb{E}[X_{2i}^2], \quad \rho_i := \frac{\mathbb{E}[X_{1i}X_{2i}]}{\sqrt{P_{1i}P_{2i}}}.$$

Define

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Lemma ("Single-Letterization")

$$|\rho| \le 1,$$

$$\sum_{i=1}^n \left(P_{1i}(1-\rho_i^2)\right) \le nP_1(1-\rho^2), \quad \textit{and}$$

$$\sum_{i=1}^n \left(P_{1i} + P_{2i} + 2\rho_i\sqrt{P_{1i}P_{2i}}\right) \le n\Big(P_1 + P_2 + 2\rho\sqrt{P_1P_2}\Big).$$

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Finally, we need to bound the probabilities. We do so using Chebyshev.

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Lemma

For any $T > 1$, choose

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\end{aligned}$$

Then, with a good choice of Q's

$$\Pr\left(\sum_{i=1}^{n} \log \frac{W(Y_i|X_{1i}, X_{2i})}{Q_{Y_i|X_{2i}}(Y_i|X_{2i})} \geq \gamma_1\right) \leq \frac{1}{T} + O(n^{-1/3})$$

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Proof Idea for the Converse Part : Completion

- Recall that

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- Choose

$$\frac{1}{T} = 1 - \varepsilon - O(n^{-1/3}) \quad \text{so} \quad \gamma_1 = n\mathsf{C}\left(\frac{P_1(1-\rho^2)}{1-\varepsilon}\right) + O(n^{2/3}).$$

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$$\log \mathsf{M}_1 \le \gamma_1 - \log^+ \left[1 - \varepsilon - \Pr\left(\sum_{i=1}^n \log \frac{W(Y_i|X_{1i}, X_{2i})}{Q_{Y_i|X_{2i}}(Y_i|X_{2i})} \ge \gamma_1 \right) \right]$$

- Probability term satisfies

$$\Pr(\cdots) \le \frac{1}{T} + O(n^{-1/3}).$$

- Choose

$$\frac{1}{T} = 1 - \varepsilon - O(n^{-1/3}) \quad \text{so} \quad \gamma_1 = n\mathsf{C}\Big(\frac{P_1(1-\rho^2)}{1-\varepsilon}\Big) + O(n^{2/3}).$$

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- By product: Second-order term is upper bounded by $O(n^{2/3})$.

Wrap Up for Fixed-Length Feedback[1]

- Generalized a result by *Ozarow (1984)* to non-vanishing $\varepsilon \in [0, 1)$

[1]"On Gaussian Channels with Feedback under Expected Power Constraints and with Non-Vanishing Error Probabilities"', L. V. Truong, S. L. Fong and V. Y. F. Tan, IEEE Transactions on Information Theory, Vol. 63, No. 3, Pages 1746–1765, Mar 2017

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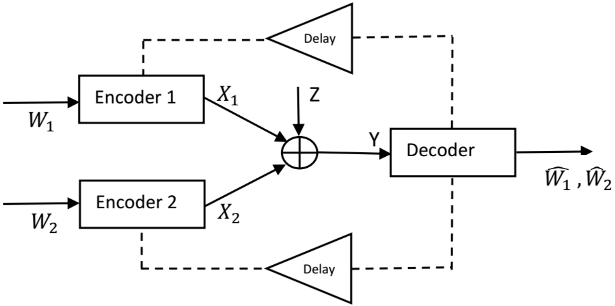
Outline

Gaussian MAC with feedback



Gaussian MAC with feedback

Consider the 2-user case. Results can be extended to the K-user setting in a straightforward way.

Stop-feedback codes for the Gaussian MAC

Let $\mathsf{M}_1, \mathsf{M}_2 \in \mathbb{N}$, $N, P_1, P_2 \in \mathbb{R}_{++}$, and $0 < \varepsilon < 1$.

An $(\mathsf{M}_1, \mathsf{M}_2, N, P_1, P_2, \varepsilon)$ stop-feedback code for the 2-user Gaussian MAC $W(y|x_1, x_2)$ is defined by:

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- Two independent random variables U_j, $j = 1, 2$.
- Each random variable is revealed to transmitter $j = 1, 2$ and the receiver before the transmission starts.
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  - (U_1, U_2) acts as common randomness used to initialize the encoders and the decoder
- Two sequences of encoders $f_n^{(1)} : \mathcal{U}_1 \times \{1, 2, \ldots, \mathsf{M}_1\} \to \mathbb{R}$ and $f_n^{(2)} : \mathcal{U}_2 \times \{1, 2, \ldots, \mathsf{M}_2\} \to \mathbb{R}$, $n \geq 1$, defining channel inputs

$$X_{j,n} = f_n^{(j)}(U_j, M_j), \qquad j = 1, 2$$

where W_j is uniform on the message set $\{1, 2, \ldots, \mathsf{M}_j\}$.

Stop-feedback codes for the Gaussian MAC

- A non-negative integer-valued random variable τ—a stopping time of the filtration $\{\sigma(U_1, U_2, Y^n)\}_{n=1}^{\infty}$—which satisfies

$$\mathbb{E}(\tau) \leq N.$$

  τ is a random blocklength whose expectation is $\leq N$.

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- A decoder which makes the final decision at time τ,

$$(\hat{M}_1, \hat{M}_2) = g_\tau(U_1, U_2, Y^\tau),$$

  and satisfies

$$\mathbb{P}\big((\hat{M}_1, \hat{M}_2) \ne (M_1, M_2)\big) \le \varepsilon.$$

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- A very limited amount of feedback is needed to send the stopping decision to the transmitters to stop their transmissions.
- Detection at the receiver is performed only at stopping time τ.

Main Result

Let $\mathsf{C}(P) = \frac{1}{2}\log(1+P)$ denote the capacity of the AWGN channel

Theorem (Truong-T. (2017))

Achievability: There exists a sequence of $(\mathsf{M}_1, \mathsf{M}_2, N, P_1, P_2, \varepsilon)$ *stop-feedback codes for the* 2*-user Gaussian MAC, for any* $(\mathsf{M}_1, \mathsf{M}_2)$ *satisfying*

$$\log \mathsf{M}_j \le \left(\frac{N}{1-\epsilon} - A\sqrt{\frac{N}{1-\varepsilon}} \right) \mathsf{C}(P_j) - \log N + O(1), \qquad j = 1,2$$

$$\log \mathsf{M}_1\mathsf{M}_2 \le \left(\frac{N}{1-\epsilon} - A\sqrt{\frac{N}{1-\varepsilon}} \right) \mathsf{C}(P_1+P_2) - \log N + O(1)$$

where $A \ge 0$ *is a constant.*

Main Results

Theorem (Truong-T. (2017))

Converse: Conversely, given any $(\mathsf{M}_1, \mathsf{M}_2, N, P_1, P_2, \varepsilon)$ *stop-feedback code for the* 2*-user Gaussian MAC, the following inequalities hold*

$$\log \mathsf{M}_j \leq \frac{N \cdot \mathsf{C}(P_j) + h_{\mathrm{b}}(\varepsilon)}{1-\varepsilon}, \qquad j = 1,2$$

$$\log \mathsf{M}_1\mathsf{M}_2 \leq \frac{N \cdot \mathsf{C}(P_1 + P_2) + h_{\mathrm{b}}(\varepsilon)}{1-\varepsilon}.$$

Corollary (Truong-T. (2017))

The ε-capacity region of the 2-user Gaussian MAC with stop-feedback, denoted $\mathcal{C}_{\mathrm{sf}}(P_1, P_2, \varepsilon)$, is the set of all rate pairs $(R_1, R_2) \in \mathbb{R}_+^2$ satisfying

$$\begin{aligned} R_j &\le \frac{\mathsf{C}(P_j)}{1-\varepsilon}, \qquad j = 1, 2 \\ R_1 + R_2 &\le \frac{\mathsf{C}(P_1 + P_2)}{1-\varepsilon}. \end{aligned}$$

Some remarks about the results

- The multiplicative factors $\frac{1}{1-\varepsilon}$ in the inequalities are due to the non-vanishing error probability regime that we study, and the use of variable-length codes with feedback

[2] S. L. Fong and V. Y. F. Tan. A proof of the strong converse theorem for Gaussian multiple access channels. IEEE Trans. Inform. Theory, 62(8):4376–4394, Aug 2016.

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- Note, in contrast, that the strong converse holds for the Gaussian MAC without feedback[2]

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- Note, in contrast, that the strong converse holds for the Gaussian MAC without feedback[2]
- However, variable-length codes generally perform better than fixed-length codes.

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VLFT Codes

An $(\mathsf{M}_1, \mathsf{M}_2, N, P_1, P_2, \varepsilon)$ variable-length feedback code with termination (VLFT) is defined as an $(\mathsf{M}_1, \mathsf{M}_2, N, P_1, P_2, \varepsilon)$ stop-feedback code except that τ is a stopping time of $\{\sigma(U_1, U_2, M_1, M_2, Y^n)\}_{n=1}^{\infty}$ and

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Theorem (Truong-T. (2017))

For VLFT codes, the ε*-capacity region,* $\mathcal{C}_{\mathrm{t}}(P_1, P_2, \varepsilon)$*, is the set of all* $(R_1, R_2) \in \mathbb{R}_+^2$ *satisfying*

$$R_j \le \frac{\mathsf{C}(P_j(1-\rho^2))}{1-\varepsilon}, \qquad j = 1, 2$$

$$R_1 + R_2 \le \frac{\mathsf{C}(P_1 + P_2 + 2\rho\sqrt{P_1 P_2})}{1-\varepsilon}$$

for some $\rho \in [0, 1]$.

Some remarks about the results

- The ε-capacity region for VLFT codes is larger than the corresponding region for fixed-length feedback codes and stop feedback codes.

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- The ε-capacity region for VLFT codes is larger than the corresponding region for fixed-length feedback codes and stop feedback codes.

- Recall for fixed-length feedback codes the ε-capacity region is given by

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$$R_1 + R_2 \le \mathsf{C}\left(\frac{P_1 + P_2 + 2\rho\sqrt{P_1P_2}}{1-\varepsilon}\right).$$

Inclusion relations

- In summary, the following relations hold:

$$\mathcal{C}_{\mathrm{no-fb}}(P_1, P_2, \varepsilon) \subsetneq \mathcal{C}_{\mathrm{fl}}(P_1, P_2, \varepsilon) \subsetneq \mathcal{C}_{\mathrm{t}}(P_1, P_2, \varepsilon)$$

$$\mathcal{R}_{\mathrm{CW}}(P_1, P_2) \subsetneq \mathcal{R}_{\mathrm{Oz}}\left(\frac{P_1}{1-\varepsilon}, \frac{P_2}{1-\varepsilon}\right) \subsetneq \frac{\mathcal{R}_{\mathrm{Oz}}(P_1, P_2)}{1-\varepsilon}$$

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- Furthermore,

$$\mathcal{C}_{\mathrm{sf}}(P_1,P_2,\varepsilon) \subsetneq \mathcal{C}_{\mathrm{t}}(P_1,P_2,\varepsilon)$$

$$\frac{\mathcal{R}_{\mathrm{CW}}(P_1,P_2)}{1-\varepsilon} \subsetneq \frac{\mathcal{R}_{\mathrm{Oz}}(P_1,P_2)}{1-\varepsilon}$$

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Proof of achievability

- First, show that $\exists$ an intermediate $\left(\mathsf{M}_1, \mathsf{M}_2, N' + o(1), P_1, P_2, \frac{1}{N'}\right)$ stop-feedback code with stopping time τ' such that

$$\log \mathsf{M}_j \le \left(N' - A\sqrt{N'}\right) \cdot \mathsf{C}(P_j) - \log N' + O(1),$$

$$\log \mathsf{M}_1 \mathsf{M}_2 \le \left(N' - A\sqrt{N'}\right) \cdot \mathsf{C}(P_1 + P_2) - \log N' + O(1),$$

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- Choose $N' \approx N/(1-\varepsilon)$ in the sequel.

- Define information densities:

$$i(x_1^n; y^n | x_2^n) := \log \frac{W_{X_1^n Y^n | X_2^n}}{P_{X_1^n | X_2^n} \times W_{Y^n | X_2^n}} (x_1^n, x_2^n, y^n).$$

[3] V. Y. F. Tan and O. Kosut. On the dispersions of three network information theory problems. IEEE Trans. Inform. Theory, 60(2):881–903, 2014.

[4] K. F. Trillingsgaard and P. Popovski. Variable-length coding for short packets over a multiple access channel with feedback. In Proc. 11th Intl. Symp. on Wireless Communications Systems, pp. 796–800, Barcelona, Spain, 2014.

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- Generate a size $(\mathsf{M}_1, \mathsf{M}_2)$ random codebook [3,4] and define stopping times as follows

$$\tau_{j,k}^{(1)} := \inf \left\{ n \geq 0 : i\left(X_1^n(j); Y^n | X_2^n(k)\right) > \gamma_1 \right\},$$
$$\tau_{j,k}^{(2)} := \inf \left\{ n \geq 0 : i\left(X_2^n(k); Y^n | X_1^n(j)\right) > \gamma_2 \right\},$$
$$\tau_{j,k}^{(3)} := \inf \left\{ n \geq 0 : i\left(X_1^n(j), X_2^n(k); Y^n\right) > \gamma_3 \right\}$$
$$\tau_{j,k} := \max \left\{ \tau_{j,k}^{(1)}, \tau_{j,k}^{(2)}, \tau_{j,k}^{(3)} \right\}$$

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- The output of the decoder is given by

$$g(Y^{\tau^*}) = \max\left\{(j,k) : \tau_{j,k} = \tau^*\right\}$$

where (j,k) is arranged in lexicographic order.

- Error probability satisfies[5,6]

$$\mathbb{P}\big(g(Y^{\tau^*}) \neq (M_1, M_2)\big) \leq (\mathsf{M}_1 - 1)(\mathsf{M}_2 - 1)\mathbb{P}(\tau' \geq \bar{\tau}^{(3)}) + (\mathsf{M}_1 - 1)\mathbb{P}(\tau' \geq \bar{\tau}^{(1)}) + (\mathsf{M}_2 - 1)\mathbb{P}(\tau' \geq \bar{\tau}^{(2)}),$$

[5] Y. Polyanskiy, H. V. Poor, and S. Verdu. Feedback in the non-asymptotic regime. IEEE Trans. Inform. Theory, 57(8):4903-4925, 2011.

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- Stopping time satisfies

$$\mathbb{E}(\tau^*) \leq \mathbb{E}(\tau'),$$

where $\tau', \bar{\tau}^{(1)}, \bar{\tau}^{(2)}, \bar{\tau}^{(3)}$ are some other stopping times.

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- Let $X_1^\infty, X_2^\infty, \bar{X}_1^\infty, \bar{X}_2^\infty, Y^\infty$ be i.i.d. infinite dimensional vectors distributed according to

$$P_{X_1}(x_1)P_{X_2}(x_2)W(y|x_1x_2)P_{X_1}(\bar{x}_1)P_{X_2}(\bar{x}_2),$$

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- Define various stopping times (adapted to filtration $\{\sigma(U_1, U_2, Y^n)\}_{n\geq 1}$):

$$\begin{aligned}
\tau^{(1)} &:= \inf\left\{n \geq 0 : i(X_1^n; Y^n|X_2^n) > \gamma_1\right\}, \\
\tau^{(3)} &:= \inf\left\{n \geq 0 : i(X_1^n, X_2^n; Y^n) > \gamma_3\right\}, \\
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\tau' &:= \max\left\{\tau^{(1)}, \tau^{(2)}, \tau^{(3)}\right\}.
\end{aligned}$$

- Using the strongly-nonlattice property of the information densities $i(X_{1,n}; Y_n | X_{2,n}), i(X_{2,n}; Y_n | X_{1,n}), i(X_{1,n}, X_{2,n}; Y_n)$ we can write the mean and variance of the stopping times as

$$\mathbb{E}[\tau^{(j)}] = N - A\sqrt{N} - G - B_j + o(1),$$
$$\mathrm{var}(\tau^{(j)}) \leq L_j N + F_j + o(1), \qquad j = 1, 2, 3,$$

for appropriately chosen $\gamma_1, \gamma_2, \gamma_3$. Here, A and G can be arbitrarily chosen constants.

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[7] A. Gut. On the moments and limit distributions of some first passage times. The Annals of Probability, 2(2):277–308, 1974.

[8] T. L. Lai and D. Siegmund. A nonlinear renewal theory with applications to sequential analysis II. The Annals of Statistics, 7(1):60–76, 1979.

- Using the strongly-nonlattice property of the information densities $i(X_{1,n}; Y_n|X_{2,n}), i(X_{2,n}; Y_n|X_{1,n}), i(X_{1,n}, X_{2,n}; Y_n)$ we can write the mean and variance of the stopping times as

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$$\mathrm{var}(\tau^{(j)}) \le L_j N + F_j + o(1), \qquad j = 1, 2, 3,$$

  for appropriately chosen $\gamma_1, \gamma_2, \gamma_3$. Here, A and G can be arbitrarily chosen constants.

- The above expressions are derived based ideas from renewal theory by Gut[7] and Lai and Siegmund[8].

[7] A. Gut. On the moments and limit distributions of some first passage times. The Annals of Probability, 2(2):277–308, 1974.

[8] T. L. Lai and D. Siegmund. A nonlinear renewal theory with applications to sequential analysis II. The Annals of Statistics, 7(1):60–76, 1979.

Lemma (Expectation of the maximum of random variables)

Let $\{(X_{1,N}, X_{2,N}, X_{3,N})\}_{N\geq 1}$ *be three sequences of random variables satisfying*

$$\mathbb{E}[X_{j,N}] = N - A\sqrt{N} - G - B_j + o(1), \quad j = 1,2,3,$$

for some constants $B_1, B_2, B_3 \in \mathbb{R}$ *and*

$$\mathrm{var}(X_{j,N}) \leq L_j N + F_j + o(1), \quad j = 1,2,3,$$

for some other constants $L_1 > 0, L_2 > 0, L_3 > 0$ *and* $F_1, F_2, F_3 \in \mathbb{R}$*.*

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Then for some constants A, G *we have*

$$\mathbb{E}[\max\{X_{1,N}, X_{2,N}, X_{3,N}\}] \leq N + o(1).$$

- Similarly, choosing a pair $(\mathsf{M}_1, \mathsf{M}_2)$ satisfying

$$\log \mathsf{M}_j \le \gamma_j - \log(3N'), \qquad j = 1, 2,$$
$$\log \mathsf{M}_1 \mathsf{M}_2 \le \gamma_3 - \log(3N'),$$

we obtain

$$\mathbb{P}\left(g(Y^{\tau^*}) \neq (M_1, M_2)\right) \le \frac{1}{N'}.$$

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- So the error probability constraint for the intermediate code is satisfied.
- But we need to control expected power and expected blocklength.

Lemma (Generalization of Wald's equation)

Let $\{X_n\}_{n=1}^{\infty}$ be an infinite sequence of real-valued random variables, τ a non-negative integer-valued random variable. Assume that

- *X_n are all integrable (finite-mean) random variables having the same mean, $\mathbb{E}[X_n] = \mathbb{E}[X_1]$;*
- *For all n, $\mathbb{E}[X_n \cdot 1\{\tau \geq n\}] = \mathbb{E}[X_n] \cdot \mathbb{P}(\tau \geq n)$;*
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Usual Wald's equation: $\{X_n\}_n$ is a sequence of i.i.d. rvs and τ a non-negative integer-valued rv independent of $\{X_n\}_n$ . . .

- First, we observe that for $j = 1, 2$,

$$\begin{aligned}\mathbb{E}\left[\sum_{n=1}^{\tau^*} X_{j,n}^2\right] &= \mathbb{E}\left[\sum_{n=1}^{\tau^*} X_{j,n}^2 \middle| (M_1, M_2) = (1, 1)\right] \\ &\leq \mathbb{E}\left[\sum_{n=1}^{\tau_{1,1}} X_{j,n}^2 \middle| (M_1, M_2) = (1, 1)\right] \overset{\text{PPV}}{\leq} \mathbb{E}\left[\sum_{n=1}^{\tau'} X_{j,n}^2\right].\end{aligned}$$

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- Then we obtain

$$\mathbb{E}\left[\sum_{n=1}^{\tau^*} X_{j,n}^2\right] \le \mathbb{E}[\tau'] P_j$$

Then terminate the code at τ^* (create new rvs $\tilde{X}_{j,n}$) so

$$\sum_{n=1}^{\infty} \mathbb{E}\big[\tilde{X}_{j,n}^2\big] = \mathbb{E}[\tau^*] P_j$$

Choosing $N' \approx N/(1-\varepsilon)$ and using power control (using the code $\left(\mathsf{M}_1, \mathsf{M}_2, N' + o(1), P_1, P_2, \frac{1}{N'}\right)$ with probability $(1-\varepsilon)$)[9], we obtain:[10]

[9] Y. Polyanskiy, H. V. Poor, and S. Verdu. Feedback in the non-asymptotic regime. IEEE Trans. Inform. Theory, 57(8):4903–4925, 2011.

[10] L. V. Truong and V. Y. F. Tan. On AWGN channels and Gaussian MACs with variable-length feedback. Submitted to IEEE Trans. Inform. Theory, 2016, revised April 2017. arXiv:1609.00594

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Lemma (Truong-T. (2017))

There exists an $(\mathsf{M}_1, \mathsf{M}_2, N + o(1), P_1, P_2, \varepsilon)$ *stop-feedback code for the* 2*-user Gaussian MAC, for any* $\mathsf{M}_1, \mathsf{M}_2$ *satisfying*

$$\log \mathsf{M}_j \le \left(\frac{N}{1-\varepsilon} - A\sqrt{\frac{N}{1-\varepsilon}} \right) \cdot \mathsf{C}(P_j) - \log N + O(1), \quad j = 1, 2$$

$$\log \mathsf{M}_1 \mathsf{M}_2 \le \left(\frac{N}{1-\varepsilon} - A\sqrt{\frac{N}{1-\varepsilon}} \right) \cdot \mathsf{C}(P_1 + P_2) - \log N + O(1)$$

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Outline

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- Future work: (i) Second-order; (ii) Replace $\mathbb{E}[\tau] \le N$ with the probabilistic constraint $\min\{n \in \mathbb{Z}^+ : \Pr(\tau > n) \le \varepsilon_{\mathrm{d}}\} \le N$ *[Altuğ, Poor and Verdú (2015)]*