On Wyner's, Exact & ∞-Rényi Common Informations: Settling a Conjecture by Kumar, Li and El Gamal

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Joint work with **Lei Yu**
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Outline

1. Background
2. Main Results
3. Proof Ideas
4. Follow-Up Work and Conclusions

Measures of information

- Consider two correlated memoryless sources (X^n, Y^n) i.i.d. $\sim P_{XY}$
  - The information contained in X^n is the entropy

$$H(X) := -\sum_x P_X(x) \log P_X(x);$$

  - The information contained in Y^n is the entropy $H(Y)$
  - The information contained jointly in (X^n, Y^n) is the joint entropy $H(X, Y)$
  - The mutual information between X^n and Y^n is

$$I(X;Y) = H(X) + H(Y) - H(X,Y).$$

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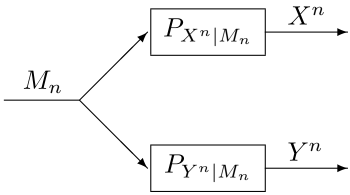
- What is "common information" between X^n and Y^n?
  - Gács-Körner Common Information
  - Wyner's Common Information
  - Exact Common Information
  - Rényi Common Information

Type 1: Wyner's Common Information



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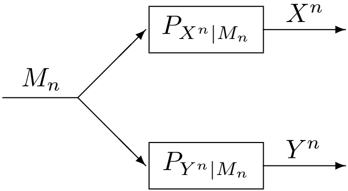

- M_n is uniformly distributed over $\mathcal{M}_n := \{1, \ldots, e^{nR}\}$
- An (n, R) synthesis code consists of
  - $P_{X^n|M_n} : \mathcal{M}_n \to \mathcal{X}^n$ and $P_{Y^n|M_n} : \mathcal{M}_n \to \mathcal{Y}^n$.
- The distribution induced by the code is

$$P_{X^nY^n}(x^n, y^n) := \frac{1}{|\mathcal{M}_n|} \sum_{m \in \mathcal{M}_n} P_{X^n|M_n}(x^n|m) P_{Y^n|M_n}(y^n|m)$$

- Desideratum: $P_{X^nY^n} \approx \pi_{XY}^n$

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- Not possible with block codes.

Type 2: Exact Common Information

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Exact Common Information

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***Abstract*—This paper introduces the notion of exact common information, which is the minimum description length of the common randomness needed for the exact distributed generation of two correlated random variables (X, Y). We introduce the quantity $G(X;Y) = \min_{X \to W \to Y} H(W)$ as a natural bound on the exact common information and study its properties and computation. We then introduce the exact common information rate, which is the minimum description rate of the common randomness for the exact generation of a 2-DMS (X, Y). We give a multiletter characterization for it as the limit $\overline{G}(X;Y) = \lim_{n\to\infty}(1/n)G(X^n;Y^n)$. While in general $\overline{G}(X;Y)$ is greater than or equal to the Wyner common information, we show that they are equal for the Symmetric Binary Erasure Source. We do not know, however, if the exact common information rate has a single letter characterization in general.**

Section II. We do not, however, know if they are equal in general.

The rest of this paper is organized as follows. In the next section we introduce the exact distributed generation problem and define the exact common information. We introduce the "common-entropy" quantity $G(X;Y) = \min_{X\to W\to Y} H(W)$ as a natural bound on the exact common information and study some of its properties. In Section III, we define the exact common information rate for a 2-DMS. We show that it is equal to the limit $\overline{G}(X;Y) = \lim_{n\to\infty}(1/n)G(X^n;Y^n)$ and that it is in general greater than or equal to the Wyner common information. One of the main results in this paper is to show that $\overline{G}(X;Y) = J(X;Y)$ for the SBES. A consequence of this

Figure: Introduced by Kumar, Li and El Gamal in ISIT 2014

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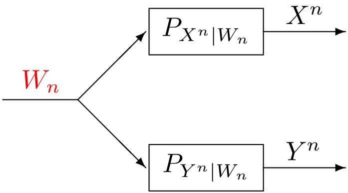

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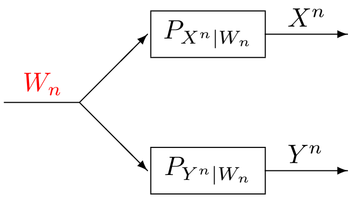

- A synthesis code $(P_{W_n}, P_{X^n|W_n}, P_{Y^n|W_n})$
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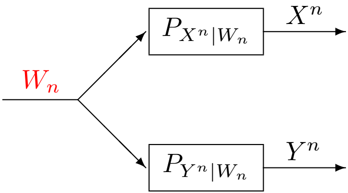

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$$f : \mathcal{W}_n \to \{0,1\}^* := \bigcup_{n\geq 1} \{0,1\}^n$$

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- Then the optimal expected codeword length $L(W_n) = \mathbb{E}[\ell(W_n)]$ satisfies

$$H(W_n) \le L(W_n) < H(W_n) + 1$$

which implies that

$$\lim_{n\to\infty} \frac{L(W_n)}{n} = \lim_{n\to\infty} \frac{H(W_n)}{n}.$$

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$$T_{\mathrm{Exact}}(\pi_{XY}) := \inf \left\{ \lim_{n\to\infty} \frac{L(W_n)}{n} \ : \ P_{X^nY^n} = \pi_{XY}^n, \forall n \ge 1 \right\}$$

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$$T_{\text{Exact}}(\pi_{XY}) = \lim_{n\to\infty} \frac{1}{n} \min_{P_W P_{X^n|W} P_{Y^n|W} : P_{X^nY^n} = \pi_{XY}^n} H(W).$$

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As expected the exact common information rate is greater than or equal to the Wyner common information.

Proposition 3.

$$\bar{G}(X;Y) \ge J(X;Y).$$

In the following section, we show that they are equal for the SBES in Example 1. We do not know if this is the case in general, however.

Excerpt from KLE (2014)

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- P_X, Q_X are two discrete distributions on the same alphabet $\mathcal{X}$.
- Rényi divergence of order $\alpha \geq 0$ is defined as

$$D_\alpha(P_X \| Q_X) := \frac{1}{\alpha - 1} \log \sum_{x \in \mathcal{X}} P_X(x) \left(\frac{P_X(x)}{Q_X(x)} \right)^{\alpha - 1}$$

  - $\alpha = 1$:

$$D_1(P_X \| Q_X) := \lim_{\alpha \to 1} D_\alpha(P_X \| Q_X) = D(P_X \| Q_X)$$

  - $\alpha = \infty$:

$$D_\infty(P_X \| Q_X) := \lim_{\alpha \to \infty} D_\alpha(P_X \| Q_X) = \log \sup_{x \in \mathrm{supp}(P_X)} \frac{P_X(x)}{Q_X(x)}$$

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- Recall that in Wyner's CI, the $D(P_X \| Q_X)$ is used as the measure
- Define the α-Rényi common information as

$$T_\alpha(\pi_{XY}) := \inf \{R : \ D_\alpha(P_{X^n Y^n} \| \pi_{XY}^n) \to 0\}$$
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- α-Rényi CI is important in building a bridge between Exact CI and Wyner's CI
  - Wyner's CI = 1-Rényi CI —— by definitions
    (In Wyner's CI, $\frac{1}{n}D$ and D do not change $C_{\mathsf{Wyner}}(\pi_{XY})$)
  - Exact CI = ∞-Rényi CI —— we will show this later

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- For doubly symmetric binary sources (DSBSes), we show that the upper and lower bounds coincide
  - Completely characterized
- Interestingly, for such sources, exact and ∞-Rényi CIs are strictly larger than Wyner's
  - This answers the open problem posed by KLE
- We extend these results to other sources, including Gaussian sources and show an improvement over Li and El Gamal's 2017 paper "Distributed simulation of continuous random variables".

Outline

Equivalence Between ∞-Rényi CI and Exact CI

Theorem

For a bivariate source π_{XY} on a finite alphabet,

$$T_{\text{Exact}}(\pi_{XY}) = T_{\infty}(\pi_{XY}).$$

Single-letter Bounds I

Theorem

$$\Gamma^{\mathrm{LB}}(\pi_{XY}) \le T_\infty(\pi_{XY}) = T_{\mathrm{Exact}}(\pi_{XY}) \le \Gamma^{\mathrm{UB}}(\pi_{XY}),$$

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For Γ^{LB}, if $Q_{WW'} \Leftarrow P_W(w)1\{w' = w\}$, then $\Gamma^{\mathrm{LB}} \Rightarrow \Gamma^{\mathrm{UB}}$. Hence

$$\Gamma^{\mathrm{UB}} \geq \Gamma^{\mathrm{LB}}$$

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$$\begin{aligned} & -H_\pi(XY|W) + \sum_w P_W^{(\pi)}(w) \sum_{x,y} P^{(\pi)}_{X|W=w}(x) P^{(\pi)}_{Y|W=w}(y) \log \frac{1}{\pi(x,y)} \\ &= -H_\pi(XY|W) + H_\pi(W) = I_\pi(XY;W) \end{aligned}$$

Hence

$$\Gamma^{\mathrm{UB}} \geq C_{\mathsf{Wyner}}$$

Doubly Symmetric Binary Source (DSBS)

- Consider (X, Y) such that $X \sim \mathrm{Bern}(\frac{1}{2})$ and $Y = X \oplus E$ with $E \sim \mathrm{Bern}(p), p \in (0, \frac{1}{2})$ independent of X

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Theorem (Evaluation of Upper and Lower Bounds for DSBS(p))

For a DSBS (X, Y)*,*

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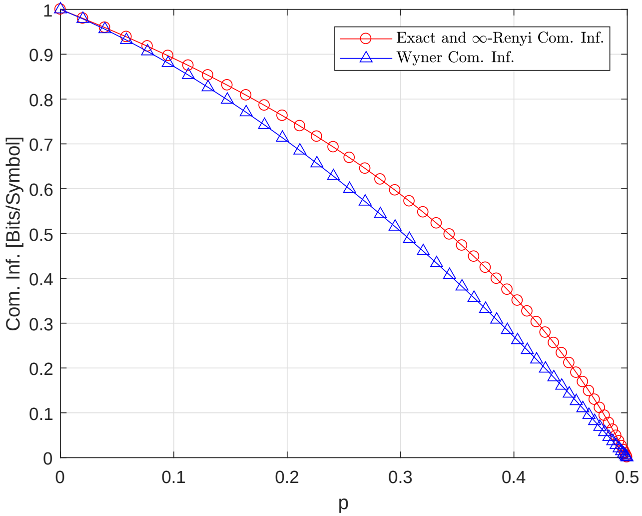
- KLE also considered Symmetric Binary Erasure Source (SBES), for which, they showed $T_{\mathrm{Exact}}(\pi_{XY}) = C_{\mathrm{Wyner}}(\pi_{XY})$

Numerical Results — DSBS

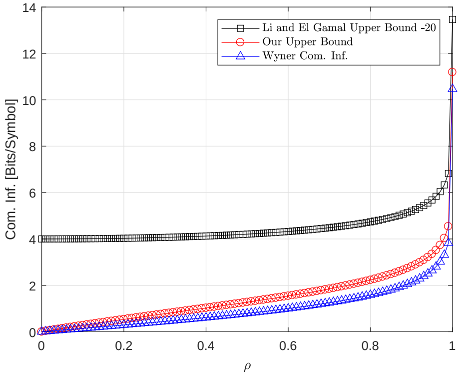


Gaussian source with Corr. Coef. $\rho \in [0,1)$

Theorem

$$\frac{1}{2}\log\left[\frac{1+\rho}{1-\rho}\right] \le T_\infty(\pi_{XY}) = T_{\text{Exact}}(\pi_{XY}) \le \frac{1}{2}\log\left[\frac{1+\rho}{1-\rho}\right] + \frac{\rho}{1+\rho}.$$

The gap $\frac{\rho}{1+\rho} \le 0.5$ nats/symbol or 0.72 bits/symbol



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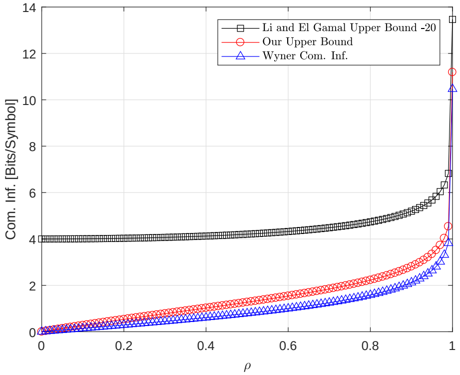

$$\pi_{XY} = \mathcal{N}\left(\begin{bmatrix}0\\0\end{bmatrix}, \begin{bmatrix}1 & \rho\\ \rho & 1\end{bmatrix}\right)$$

$$T_{\mathrm{Exact}}^{(\mathrm{Li-El\,Gamal})}(\pi_{XY}) \le \frac{1}{2}\log\frac{1}{1-\rho^2} + 24\log 2.$$

Outline

Proof Sketch

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  - $\exists$ rate-R exact CI code $\iff$ $\exists$ rate-R ∞-Rényi CI code

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- Step 3: prove the converse part (lower bound) for ∞-Rényi CI

Step 1: Equivalence: $\Longrightarrow$

Lemma (Vellambi-Kliewer (2016))

$\exists$ *rate-*R ∞*-Rényi CI code* $\Longrightarrow$ $\exists$ *rate-*R *exact CI code*

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then obviously, $\widehat{P}_{X^nY^n}(x^n, y^n)$ is a distribution

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- Hence π_{XY}^n can be written as a mixture distribution

$$\pi_{XY}^n(x^n, y^n) = e^{-\epsilon} P_{X^nY^n}(x^n, y^n) + \left(1 - e^{-\epsilon}\right) \widehat{P}_{X^nY^n}(x^n, y^n)$$

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- A time-sharing variable-length scheme:
  - The encoder first generates $U \sim \mathrm{Bern}(e^{-\epsilon})$, and transmits it to two generators using 1 bit
  - If $U = 1$, then the encoder and two generators use the rate-R ∞-Rényi CI code to generate $P_{X^nY^n}$
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- The induced distribution is π_{XY}^{n} exactly
- The total code rate

$$\leq \frac{1}{n}+e^{-\epsilon} R+\left(1-e^{-\epsilon}\right) \log |\mathcal{X}||\mathcal{Y}| \rightarrow R$$

as $n \rightarrow \infty, \epsilon \rightarrow 0$

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  - Then by assumption

$$D_\infty(P_{f(M)} \| P^n_{W_k}) \xrightarrow{n\to\infty} 0 \qquad \text{if} \qquad R > \frac{1}{k}H(P_{W_k}).$$

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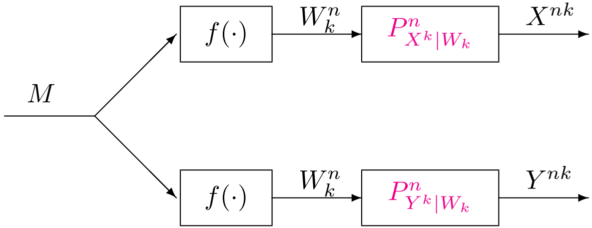

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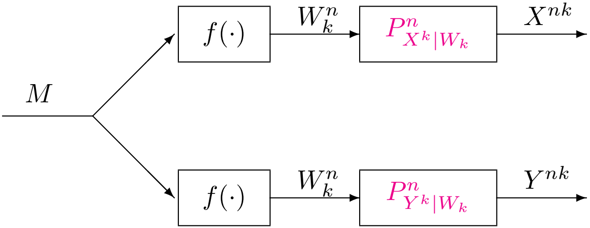

- For the given channel $P^n_{X^k|W_k} P^n_{Y^k|W_k}$,

$$P_W^n \longrightarrow P^n_{X^k|W_k} P^n_{Y^k|W_k} \longrightarrow \pi^{kn}_{XY}$$

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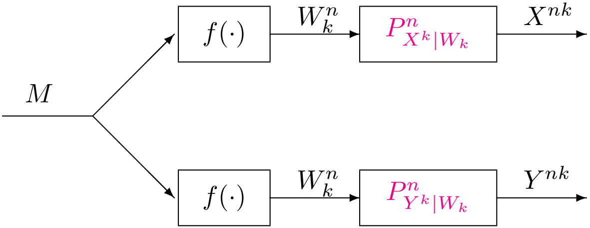

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- By the data processing inequality (DPI) for Rényi divergence,

$$D_\infty(P_{X^{kn}Y^{kn}}\|\pi^{kn}_{XY}) \le D_\infty(P_{f(M)}\|P^n_{W_k}) \overset{n\to\infty}{\longrightarrow} 0$$

Step 2: Achievability for $T_\infty(\pi_{XY})$

- For $0 < \epsilon' < \epsilon \le 1$, define the truncated product distributions

$$Q_{W^n}(w^n) \propto P_W^n(w^n) 1\left\{w^n \in \mathcal{T}_{\epsilon'}^{(n)}(P_W)\right\},$$

$$Q_{X^n|W^n}(x^n|w^n) \propto P_{X|W}^n(x^n|w^n) 1\left\{x^n \in \mathcal{T}_{\epsilon}^{(n)}\left(P_{X|W}|w^n\right)\right\},$$

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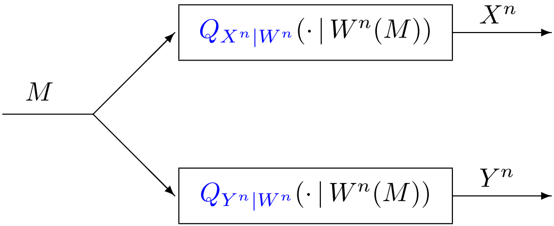
- Traditional random code, but generated according to these truncated product distributions



$\mathcal{C} = \{W^n(m)\}_{m\in\mathcal{M}}$ with $W^n(m) \sim Q_{W^n}$

Step 2: Achievability for $T_\infty(\pi_{XY})$

By using Union Bound and Bernstein's inequality, we show that for such a code, if $R \ge \Gamma^{\mathrm{UB}}(\pi_{XY})$, then

$$\max_{(x^n,y^n)\in\mathrm{supp}(P_{X^nY^n})} \frac{P_{X^nY^n}(x^n,y^n)}{\pi_{XY}^n(x^n,y^n)} \le 1+o(1)$$

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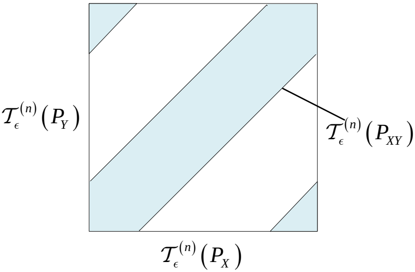

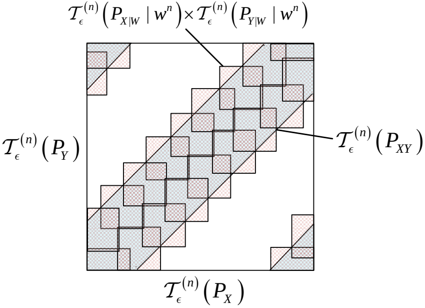

Step 3: Converse for $T_\infty(\pi_{XY})$

- Key steps in Converse Part:

$$nR \geq \max_{m} \max_{x^n, y^n} \log \frac{P_{X^n|M}(x^n|m) P_{Y^n|M}(y^n|m)}{\pi_{XY}^n(x^n, y^n)} \qquad <\!\!-\!\!-\!\!-\!\!-\!\!> \qquad \text{definition of } D_\infty$$

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$$\ge \sum_m P_M(m) \max_{Q_{X^nY^n|M}\in C(P_{X^n|M},P_{Y^n|M})} \sum_{x^n,y^n} Q(x^n,y^n|m)$$

$$\times \log \frac{P_{X^n|M}(x^n|m)P_{Y^n|M}(y^n|m)}{\pi_{XY}^n(x^n,y^n)} \quad <\!\!-\!\!-\!\!-\!\!-\!\!> \quad \max \ge \text{average}$$

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$$nR \ge \max_m \max_{x^n,y^n} \log \frac{P_{X^n|M}(x^n|m)P_{Y^n|M}(y^n|m)}{\pi_{XY}^n(x^n,y^n)} \quad <\!\!-\!\!-\!\!-\!\!-\!\!> \quad \text{definition of } D_\infty$$

$$\ge \sum_m P_M(m) \max_{Q_{X^nY^n|M}\in C(P_{X^n|M},P_{Y^n|M})} \sum_{x^n,y^n} Q(x^n,y^n|m)$$

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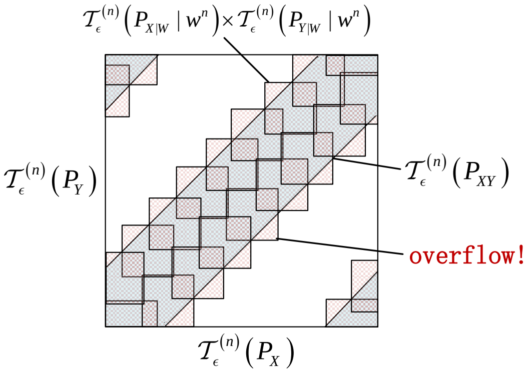
- Key steps in Single-letterization:
  - $-H(X^n|W) - H(Y^n|W)$ by traditional method (chain rule)
  - for the last term,

$$\max_{Q_{X^nY^n|M}\in C(P_{X^n|M},P_{Y^n|M})} \geq \max_{Q_{X_iY_i|X^{i-1}Y^{i-1}M}\in C(P_{X_i|X^{i-1}M},P_{Y_i|Y^{i-1}M}),\forall i\in[1:n]}$$

Why Exact CI (or ∞-Rényi CI) $>$ Wyner's CI?



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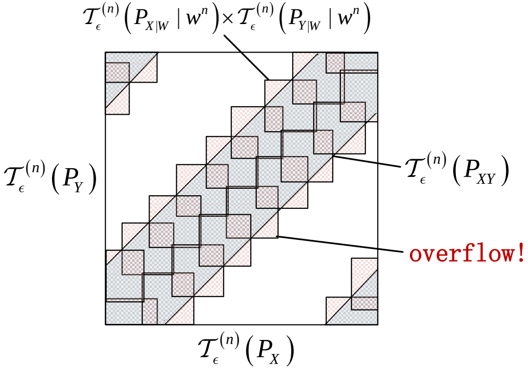

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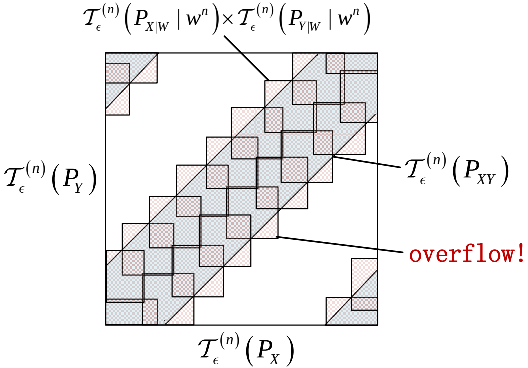

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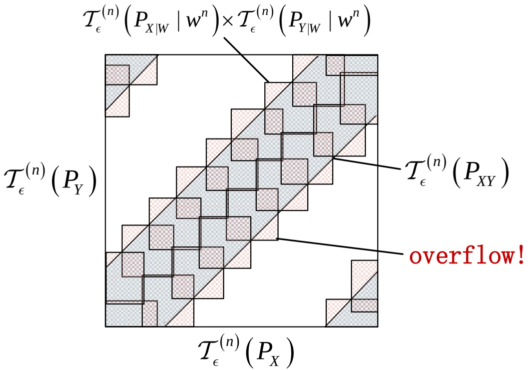

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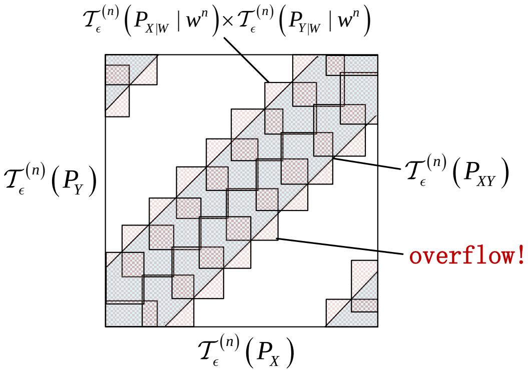

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When Exact CI (or ∞-Rényi CI) $=$ Wyner's CI?

- Sufficient condition:
  $H(X|W=w)H(Y|W=w)=0$ for each w [Vellambi-Kliewer 2016]
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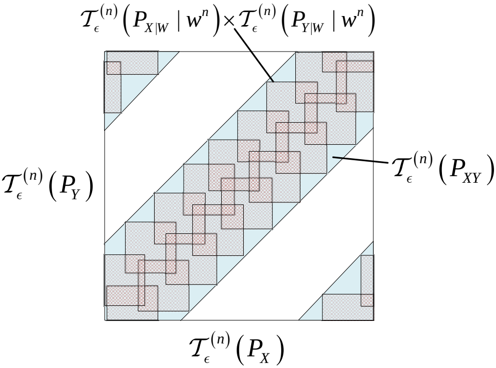

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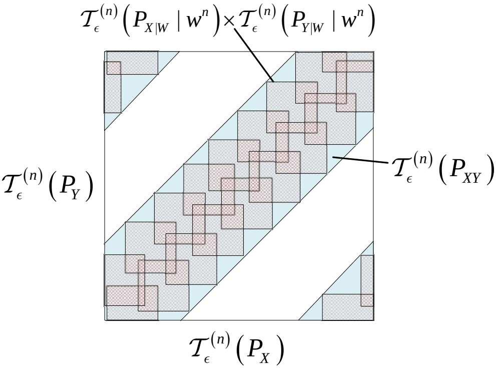

- For this case, Wyner's CI code forms a "perfect covering"

When Exact CI (or ∞-Rényi CI) $=$ Wyner's CI

Example for Sufficient Condition: $H(X|W=w)H(Y|W=w)=0$ for each w [Vellambi-Kliewer 2016]

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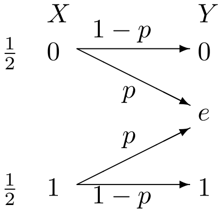
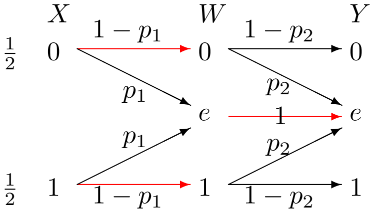
- Symmetric Binary Erasure Source (SBES)

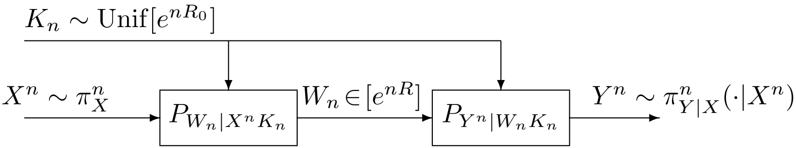




- where $(1-p_1)(1-p_2)=1-p$
- $T_\infty(\pi_{XY}) = T_{\mathrm{Exact}}(\pi_{XY}) = C_{\mathsf{Wyner}}(\pi_{XY}) = \begin{cases} 1 & p \le \frac{1}{2} \\ H(p) & p > \frac{1}{2} \end{cases}$

Outline

- How much information is required to create correlation remotely?



Exact CI is equivalent to Exact Channel Simulation

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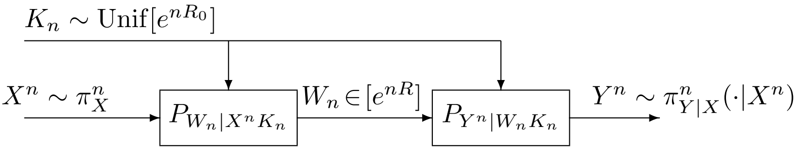

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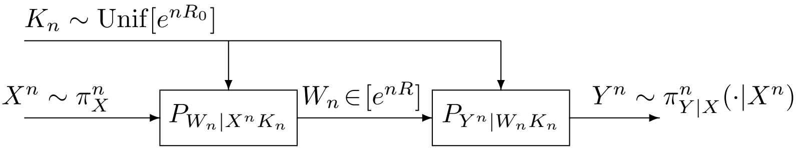

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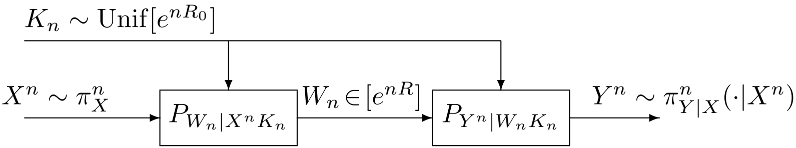

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- Sharpens a bound on the shared information rate in "Quantum Reverse Shannon Theorem" by Bennett, Devetak, Harrow, Shor, and Winter (IEEE T-IT, 2014).
  - The proof that a linear number of bits is sufficient for exact channel simulation was achieved by Harsha *et al.* (2010) and Li and El Gamal (2018).

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- We extend these results to other sources, including Gaussian sources
- L. Yu and V. Y. F. Tan, "On exact and ∞-Rényi common informations," submitted to IEEE Trans. Inf. Theory, Oct. 2018.

Thank you for your attention!

Lei Yu