Common Information and Non-Interactive Correlation Distillation

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Special thanks to **Lei Yu** (Nankai University)

2021 East Asian School on Information Theory

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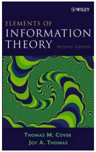
- Prerequisite: Information theory at the level of [Cover and Thomas, 2006]



Outline

Measures of Information Among Random Variables

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- As information theorists, we like operational interpretations
- Wyner's CI and Gäcs–Körner–Witsenhausen's CI are the two archetypal notions of information among RVs that admit operational interpretations.

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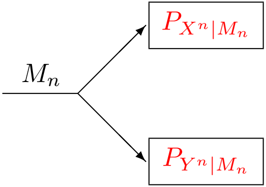

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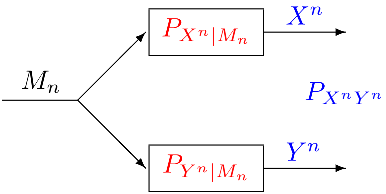

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- The distribution induced by the code $(P_{X^n|M_n}, P_{Y^n|M_n})$ is

$$P_{X^nY^n}(x^n, y^n) := \frac{1}{|\mathcal{M}_n|} \sum_{m \in \mathcal{M}_n} P_{X^n|M_n}(x^n|m) P_{Y^n|M_n}(y^n|m)$$

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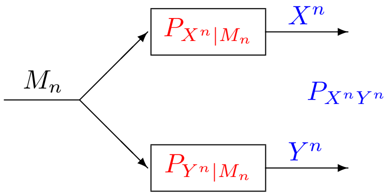

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- Desideratum:

$$P_{X^nY^n} \approx \pi_{XY}^n \qquad \text{(target distribution)}$$

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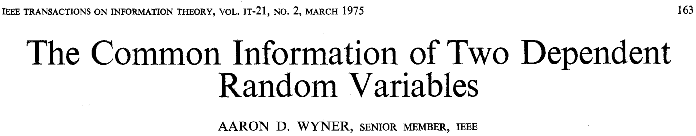
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where $C_{\mathrm{W}}(\pi_{XY})$ is named Wyner's Common Information.

Sanity Check I

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- Intuitively, we should get $H(V)$ as the common information. Do we?
- Take $W = V$, satisfies $X - W - Y$. Then

$$I(XY;W) = I(XY;V) \leq H(V) \qquad \text{so far so good...}$$

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- Minimize over $X - W - Y$ so

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Proof Idea of the Achievability Part

Lemma (Soft-covering lemma [Wyner, 1975] [Cuff, 2012])

Let $(U, W) \sim P_{UW}$ *have mutual information* $I(U;W)$. *For any*

$$R > I(U;W),$$

there exists a sequence of codebooks $\mathcal{C}_n = \{w^n(m) : m \in [2^{nR}]\}$ *such that the synthesized distribution*

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Also known as resolvability [Han and Verdú, 1993], [Hayashi, 2006], [Hayashi, 2011] and [Yu and Tan, 2019c].

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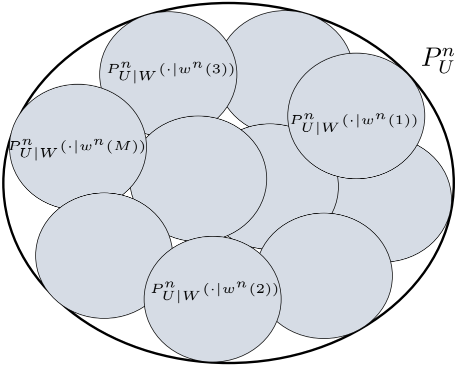

Figure: If $M = 2^{nR}$ and $R > I(U;W)$, then $\frac{1}{n} D(P_{U^n} \| P_U^n) \to 0$.

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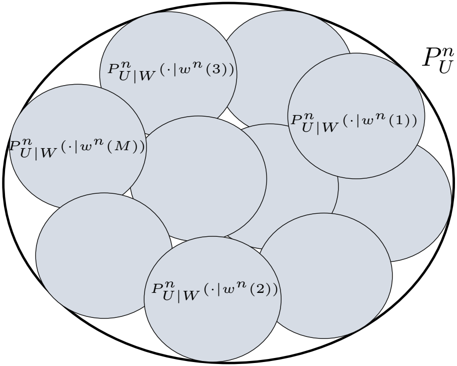
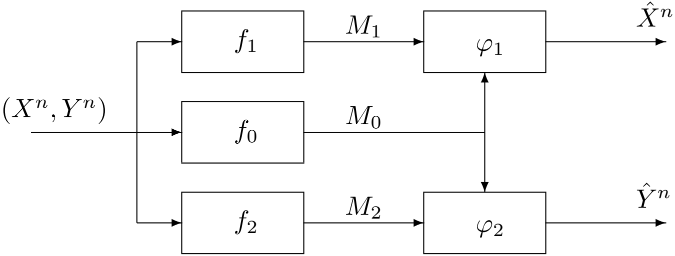


Figure: If $M = 2^{nR}$ and $R > I(U;W)$, then $\frac{1}{n}D(P_{U^n}\|P_U^n) \to 0$.

Now take $U = (X,Y) \sim \pi_{XY}$ and note by Markovity $X - W - Y$ that

$$P_{X^n|M_n}(x^n|m)P_{Y^n|M_n}(y^n|m) = P_{U^n|W^n}(u^n|w^n(m)) \text{ and } I(W;U) = I(W;XY).$$

Alternative Interpretation of Wyner's Common Information



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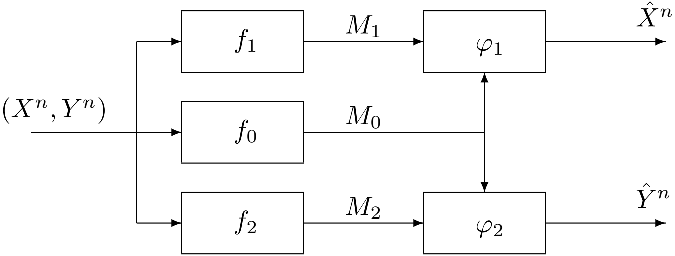

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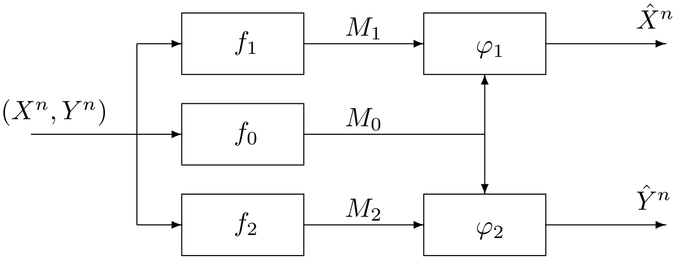

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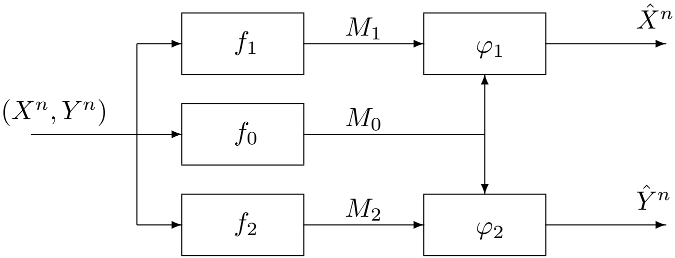

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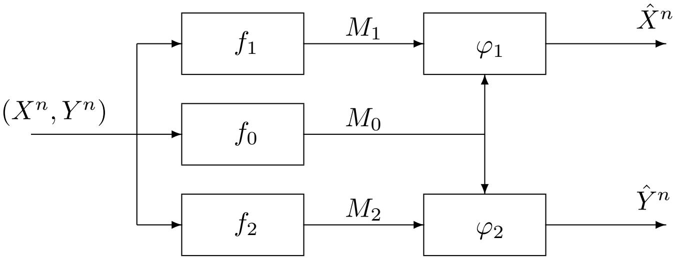

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- The probability of error of the code is

$$\Pr\big((\varphi_1(M_0, M_1), \varphi_2(M_0, M_2)) \neq (X^n, Y^n)\big).$$

where $M_i = f_i(X^n, Y^n)$ for $i = 0, 1, 2$.

Alternative Interpretation of Wyner's Common Information

Common information based on the Gray-Wyner system $T_{\mathrm{GW}}(\pi_{XY})$ for $(X,Y) \sim \pi_{XY}$

$\Longleftrightarrow$

Smallest common rate R_0 such that for all $\epsilon > 0$, there exists sequence of (n, R_0, R_1, R_2) Gray-Wyner codes $\{(f_{0,n}, f_{1,n}, f_{2,n}, \varphi_{1,n}, \varphi_{2,n})\}_{n=1}^{\infty}$ such that

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and the probability of error vanishes.

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Theorem ([Wyner, 1975])

$$T_{\mathrm{GW}}(\pi_{XY}) = C_{\mathrm{W}}(\pi_{XY})$$

Example: Doubly Symmetric Binary Source (DSBS)

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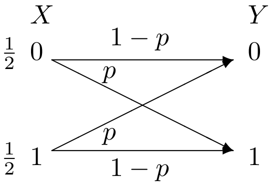

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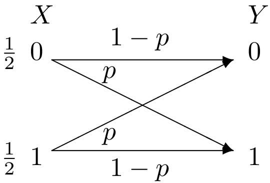

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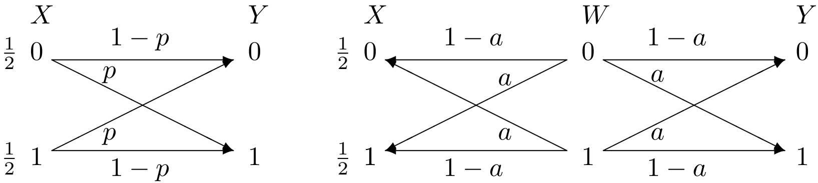

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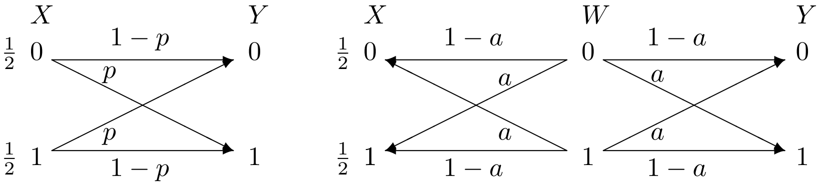
- Interpretation in terms of $X - W - Y$



- Here, $a * a = p$ and

$$a = \frac{1 - \sqrt{1-2p}}{2} \in (0, 1/2).$$

Example: DSBS

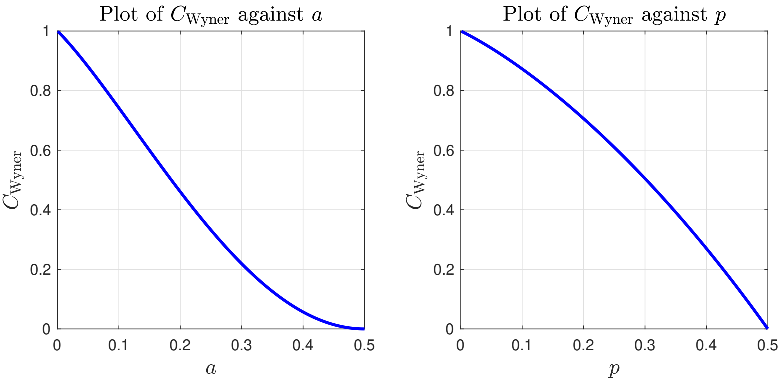


Figure: Plots of Wyner's common information for the DSBS in terms of p and a

Outline

Motivation for Alternative Measures

- Wyner used the normalized relative entropy, i.e.,

$$\inf\left\{R : \lim_{n\to\infty} \frac{D(P_{X^nY^n}\|\pi_{XY}^n)}{n} = 0\right\} = C_{\mathrm{W}}(\pi_{XY}) = \min_{X-W-Y} I(W;XY).$$

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- What if we do not normalize?

$$\tilde{T}(\pi_{XY}) := \inf\left\{R : \lim_{n\to\infty} D(P_{X^nY^n}\|\pi_{XY}^n) = 0\right\} \geq C_{\mathrm{W}}(\pi_{XY}).$$

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- What if we want an even stronger measure of dependence?
- Rényi common information for orders ≥ 1 [Yu and Tan, 2018]!

$$T_{1+s}(\pi_{XY}) := \inf\left\{R : \lim_{n\to\infty} \frac{D_{1+s}(P_{X^nY^n}\|\pi_{XY}^n)}{n} = 0\right\}$$

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Rényi Common Information

- Rényi divergence

$$D_{1+s}(P\|Q) := \frac{1}{s}\log\sum_{x\in\mathrm{supp}(P)} P(x)\left(\frac{P(x)}{Q(x)}\right)^s \quad s\in[-1,\infty)$$

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- And for a fixed order $1 + s \in [0, \infty]$,

$$T_{1+s}(\pi_{XY}) \le \tilde{T}_{1+s}(\pi_{XY}).$$

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  for some joint sources π_{XY}.
- But let's soldier on and tackle the Rényi common information for now.

Rényi Common Information: The Weaker Case

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Let's start with a simple exercise. Consider the case $s \in (-1, 0]$ in which

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Theorem ([Yu and Tan, 2018] [Yu and Tan, 2020a])

For Rényi orders in $(0, 1]$ (i.e., $s \in (-1, 0]$),

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Theorem ([Yu and Tan, 2018])

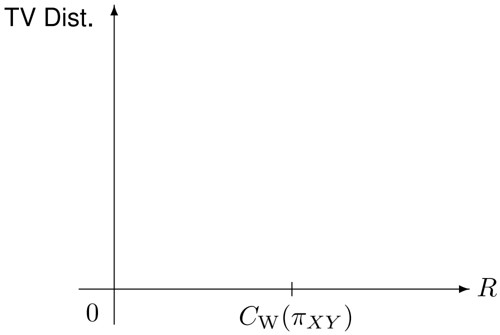
For any $\varepsilon \in [0, 1)$,

$$T_{\varepsilon}^{\mathrm{TV}}(\pi_{XY}) = C_{\mathrm{W}}(\pi_{XY}), \qquad \textit{(Strong converse)}$$

where $T_{\varepsilon}^{\mathrm{TV}}(\pi_{XY})$ *is the minimum simulation rate required to ensure*

$$\limsup_{n \to \infty} |P_{X^n Y^n} - \pi_{XY}^n| \leq \varepsilon.$$

Total Variation Common Information



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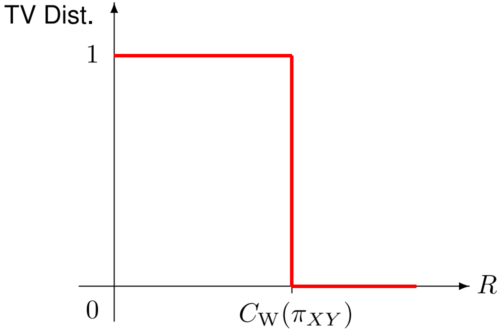

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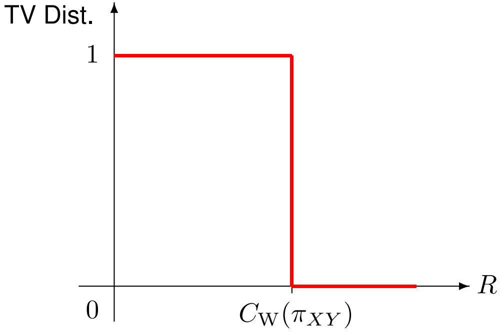

In fact, we have an exponential strong converse, i.e., if $R < C_{\mathrm{W}}(\pi_{XY})$,

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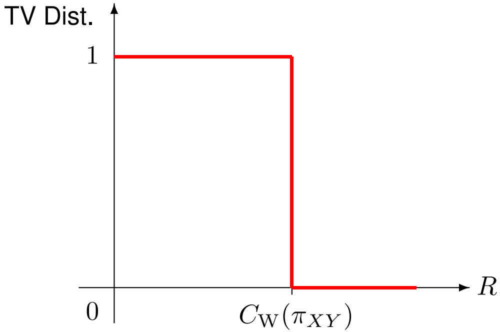

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Amenable to second-order?

Total Variation Common Information

- Achievability part follows from the soft-covering lemma.

$$\text{If} \quad R > I(XY;W) \quad \text{then} \quad \lim_{n\to\infty} |P_{X^nY^n} - \pi_{XY}^n| = 0.$$

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- Converse requires a very cool information spectrum, single-letterization idea from [Oohama, 2018].

entropy MDPI

Article

Exponential Strong Converse for Source Coding with Side Information at the Decoder †

Yasutada Oohama

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† This paper is an extended version of our paper published in 2016 International Symposium on Information Theory and Its Applications, Monterey, CA, USA, 6–9 November 2016; pp. 171–175.

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check for updates

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IEEE TRANSACTIONS ON INFORMATION THEORY, VOL. 62, NO. 1, JANUARY 2016 23

On the Rényi Divergence, Joint Range of Relative Entropies, and a Channel Coding Theorem

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For any $s \in (-1, 0]$,

$$\inf_{P_X, Q_X : |P_X - Q_X| \ge \epsilon} D_{1+s}(P_X \| Q_X) = \inf_{q \in [0, 1-\epsilon]} d_{1+s}(q + \epsilon \| q)$$

and

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- If $R < C_{\mathrm{W}}(\pi_{XY})$, exponential strong converse to TV CI says

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- Thus, if $R < C_{\mathrm{W}}(\pi_{XY})$

$$\frac{1}{n} \inf_{P_X, Q_X : |P_X - Q_X| \ge \epsilon} D_{1+s}(P_X \| Q_X) \ge \frac{1}{n} \left[\min\left\{1, \frac{1+s}{s}\right\} nE + \frac{1}{s} \log 2 \right]^+$$

  and the normalized Rényi divergence cannot vanish.

Rényi CI: The Stronger Case $s \in (0,1] \cup \{\infty\}$

- For $s \in (0,1] \cup \{\infty\}$,
$$C_{\mathrm{W}}(\pi_{XY}) \le T_{1+s}(\pi_{XY}).$$

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Definition

The maximal cross entropy w.r.t. $(X,Y) \sim \pi_{XY}$ over couplings of (P_X, P_Y) is

$$\mathsf{H}_\infty(P_X, P_Y \| \pi_{XY}) := \max_{Q_{XY} \in \mathcal{C}(P_X, P_Y)} \sum_{x,y} Q_{XY}(x,y) \log \frac{1}{\pi_{XY}(x,y)},$$

where

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- $\mathrm{H}_\infty(\pi_X, \pi_Y \| \pi_{XY}) \ge H_\pi(X;Y)$ with equality iff $\pi_{XY} = \pi_X \pi_Y$.

Intuition for the Maximal Cross Entropy

- Take a sequence of n-types $T_X^{(n)} \in \mathcal{P}_n(\mathcal{X})$ and $T_Y^{(n)} \in \mathcal{P}_n(\mathcal{Y})$.

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- So $\mathsf{H}_\infty(P_X, P_Y \| \pi_{XY})$ is the exponential decay rate of this probability.

Upper and Lower Pseudo Common Informations

Definition

The upper pseudo-common information is

$$\overline{\Gamma}_\infty(\pi_{XY}) := \min_{\substack{P_W P_{X|W} P_{Y|W}:\\ P_{XY}=\pi_{XY}}} -H(XY|W) + \mathsf{E}_{P_W}\left[\mathsf{H}_\infty(P_{X|W}, P_{Y|W} \| \pi_{XY})\right]$$

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Contrast to Wyner's common information

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Rényi Common Information of order ∞

Theorem ([Yu and Tan, 2020a] [Yu and Tan, 2020c])

The order-∞ Rényi common information admits the following single-letter bounds

$$\tilde{T}_\infty(\pi_{XY}) \geq T_\infty(\pi_{XY}) \geq \max\{\underline{\Gamma}_\infty(\pi_{XY}), C_{\mathrm{W}}(\pi_{XY})\}$$

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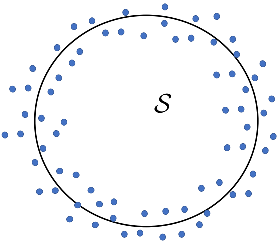

Product distribution

$$P_W^n(w^n) = \prod_{i=1}^n P_W(w_i)$$

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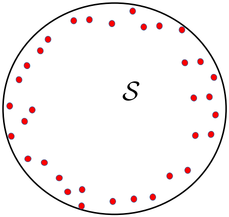

Truncated product distribution

$$P_{W^n}(w^n) \propto \left(\prod_{i=1}^{n} P_W(w_i)\right) \mathbb{1}\{w^n \in \mathcal{S}\}$$

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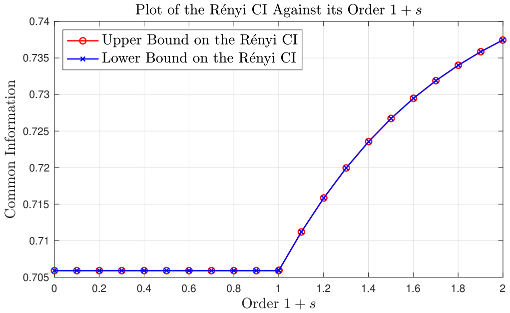

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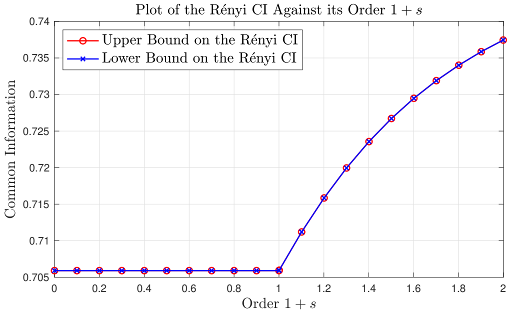

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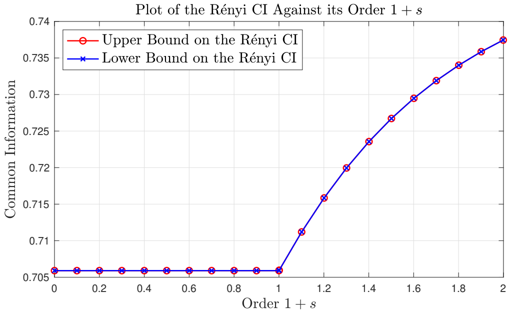

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Does this have more profound implications?

Outline

Exact Common Information?

- In the distributed source simulation problem à la Wyner, we mandated that

$$\frac{1}{n} D(P_{X^n Y^n} \| \pi_{XY}^n) \to 0.$$

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- In come [Kumar et al., 2014], who introduced

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Exact Common Information

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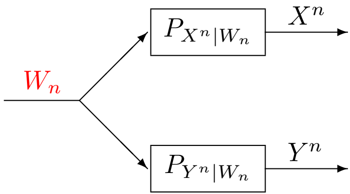

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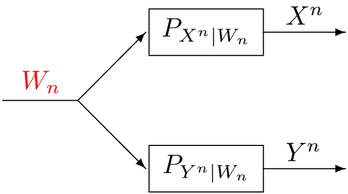

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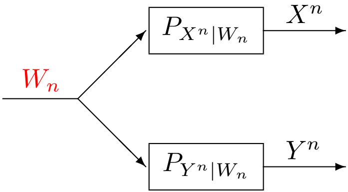

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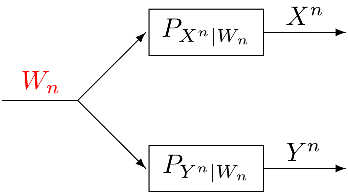

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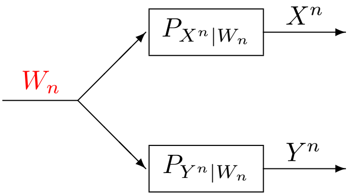

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- Let the length of W_n be $\ell(W_n)$.
- Then, by Shannon's zero-error compression theorem, the optimal expected codeword length $L(W_n) = \mathbb{E}[\ell(W_n)]$ satisfies

$$H(W_n) \leq L(W_n) < H(W_n) + 1$$

which implies that

$$\lim_{n\to\infty} \frac{L(W_n)}{n} = \lim_{n\to\infty} \frac{H(W_n)}{n}.$$

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The exact common information is defined as

$$T_{\mathrm{Ex}}(\pi_{XY}) := \inf\left\{\lim_{n\to\infty}\frac{L(W_n)}{n} : P_{X^nY^n} = \pi_{XY}^n \text{ for some } n \geq 1\right\}$$

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As expected the exact common information rate is greater than or equal to the Wyner common information.

Proposition 3.

$$\overline{G}(X;Y) \geq J(X;Y).$$

In the following section, we show that they are equal for the SBES in Example 1. We do not know if this is the case in general, however.

From [Kumar et al., 2014]

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the exact common information rate. While this multiletter characterization is in general greater than or equal to the Wyner common information, we showed that they are equal for the SBES. The main open question is whether the exact common information rate has a single letter characterization in general. Is it always equal to the Wyner common information? Is there an example 2-DMS for which the exact common information rate is strictly larger than the Wyner common information? It would also be interesting to further explore the application to machine learning.

From [Kumar et al., 2014]

Surprising Equivalence: ∞-Rényi CI and Exact CI

Theorem ([Yu and Tan, 2020c])

For a bivariate source π_{XY} on a finite alphabet,

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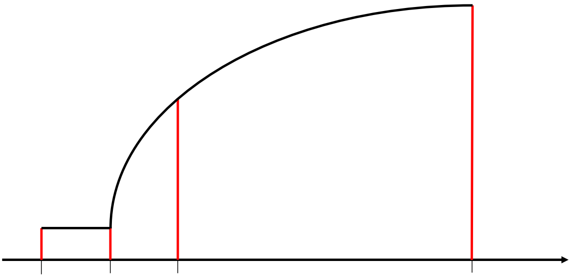

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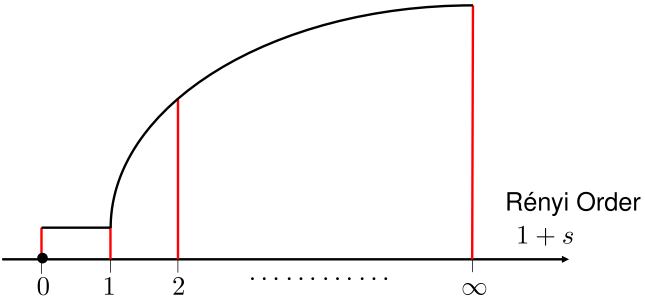

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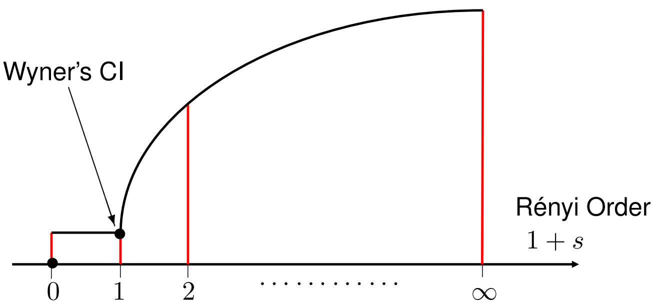

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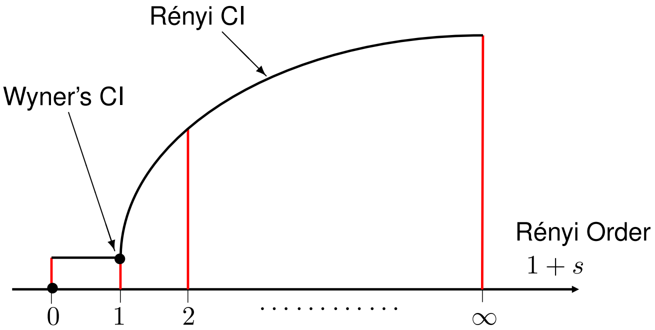

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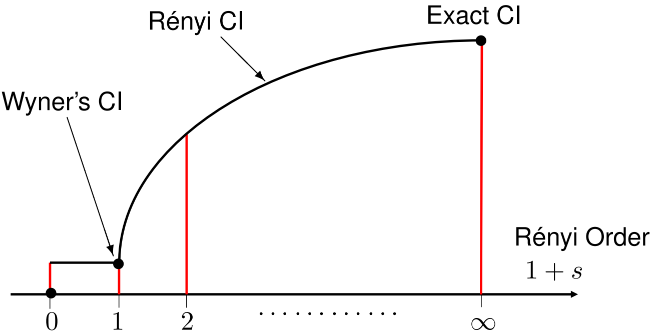

Proof of $\Longrightarrow$ Part of Equivalence Theorem

Lemma ([Kumar et al., 2014], [Vellambi and Kliewer, 2016])

$\exists$ *rate-*R ∞*-Rényi CI code* $\Longrightarrow$ $\exists$ *rate-*R *Exact CI code*

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- $\exists$ rate-R ∞-Rényi CI code

$$D_\infty(P_{X^nY^n}\|\pi_{XY}^n) < \epsilon \quad \Longrightarrow \quad P_{X^nY^n}(x^n, y^n) < 2^\epsilon \pi_{XY}^n(x^n, y^n)$$

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then obviously, $\widehat{P}_{X^nY^n}(x^n, y^n)$ is a valid distribution.

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- Hence π_{XY}^n can be written as a mixture distribution

$$\pi_{XY}^n(x^n, y^n) = 2^{-\epsilon} P_{X^nY^n}(x^n, y^n) + \left(1 - 2^{-\epsilon}\right) \widehat{P}_{X^nY^n}(x^n, y^n)$$

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- A time-sharing variable-length scheme:
  - The encoder first generates $U \sim \mathrm{Bern}(2^{-\epsilon})$, and transmits it to two generators using 1 bit
  - If $U = 1$, then the encoder and two generators use the rate-R ∞-Rényi CI code to generate $P_{X^nY^n}$
  - If $U = 0$, then the encoder generates $(X^n, Y^n) \sim \widehat{P}_{X^nY^n}$, and compresses it with rate $\log(|\mathcal{X}||\mathcal{Y}|)$ to generate $\widehat{P}_{X^nY^n}$

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- The induced distribution is π_{XY}^n exactly
- The total code rate

$$\leq \frac{1}{n} + 2^{-\epsilon} R + \left(1 - 2^{-\epsilon}\right) \log(|\mathcal{X}||\mathcal{Y}|) \to R$$

as $n \to \infty, \epsilon \to 0$

Proof of $\Longleftarrow$ Part of Equivalence Theorem

Lemma

$\exists$ *rate-*R ∞*-Rényi CI code* $\Longleftarrow$ $\exists$ *rate-*R *Exact CI code*

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$\exists$ rate-R ∞-Rényi CI code $\Longleftarrow$ $\exists$ rate-R Exact CI code

- Let $\{(P_{W_k}, P_{X^k|W_k}, P_{Y^k|W_k})\}_{k\in\mathbb{N}}$ be rate-R exact CI codes such that

$$\lim_{k\to\infty} \frac{1}{k} H(P_{W_k}) = R$$

but W_k is not uniform. 

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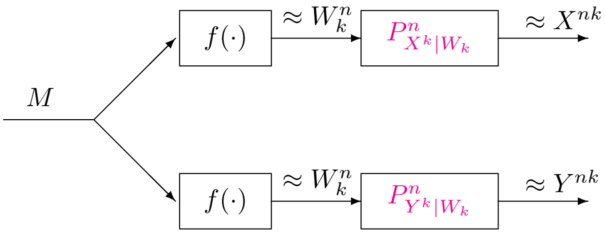
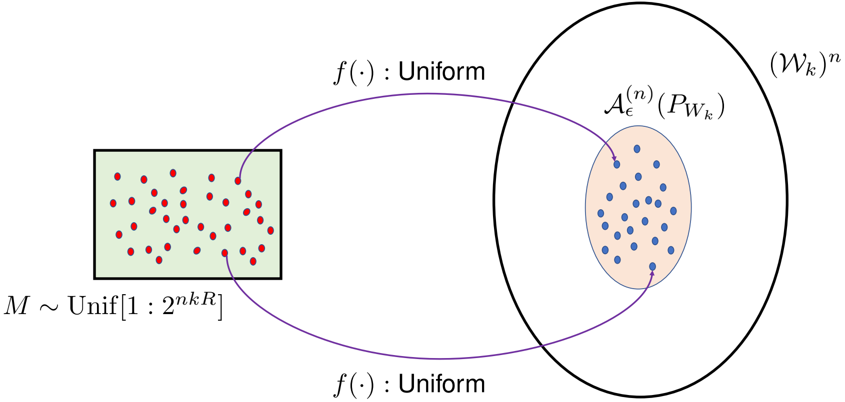
  but W_k is not uniform.
- Simulate W_k^n using two Rényi source resolvability codes!



Proof of $\Longleftarrow$ Part of Equivalence Theorem



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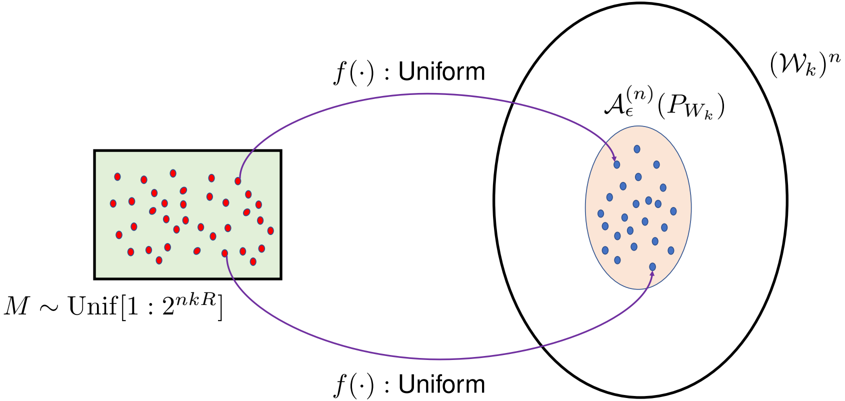

Succeed in the sense of $D_\infty(P_{f(M)} \| P_{W_k}^n) \to 0$ if [Yu and Tan, 2019d]

$$R > \frac{1}{k} H(P_{W_k})$$

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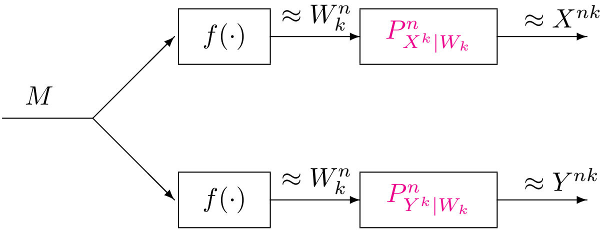

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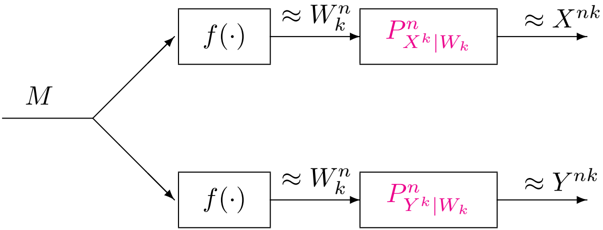

- For the given stochastic kernel (channel) $P^n_{X^k|W_k} P^n_{Y^k|W_k}$,

$$P_W^n \longrightarrow P^n_{X^k|W_k} P^n_{Y^k|W_k} \longrightarrow \pi^{kn}_{XY}$$

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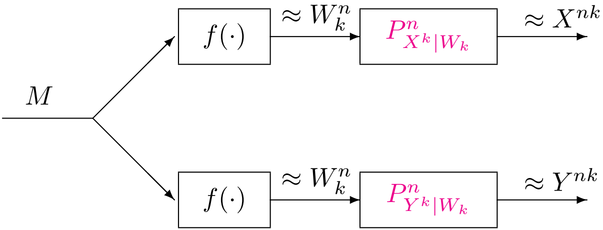

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- By the data processing inequality (DPI) for Rényi divergence,

$$D_\infty(P_{X^{kn} Y^{kn}} \| \pi^{kn}_{XY}) \le D_\infty(P_{f(M)} \| P^n_{W_k}) \overset{n\to\infty}{\longrightarrow} 0$$

Combining with Single-Letter Bounds from Rényi CI

Theorem ([Yu and Tan, 2020c])

For $(X, Y) \sim \pi_{XY}$ *on a finite alphabet,*

$$\underline{\Gamma}_\infty(\pi_{XY}) \le T_{\mathrm{Ex}}(\pi_{XY}) = \tilde{T}_\infty(\pi_{XY}) \le \overline{\Gamma}_\infty(\pi_{XY}).$$

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- Gone from a multi-letter expression by [Kumar et al., 2014]

$$\lim_{n\to\infty} \frac{1}{n} \min_{\substack{P_{W_n} P_{X^n|W_n} P_{Y^n|W_n} : \\ P_{X^n Y^n} = \pi_{XY}^n}} H(W_n)$$

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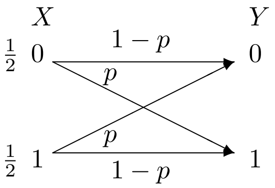
  to single-letter bounds.

- Presumably the bounds are more amenable to numerical evaluation?

Revisiting the DBSS



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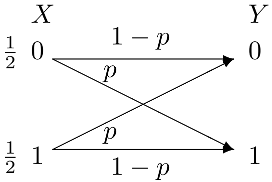

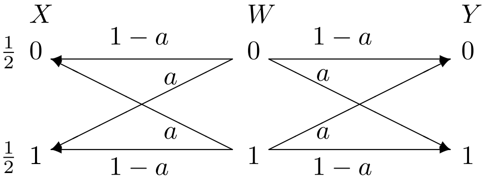

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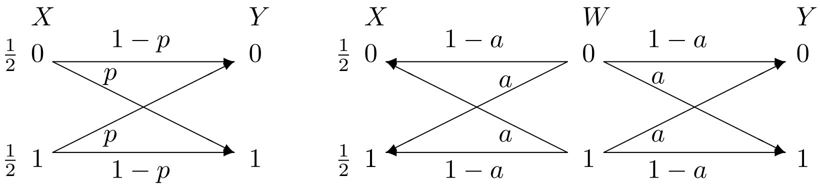


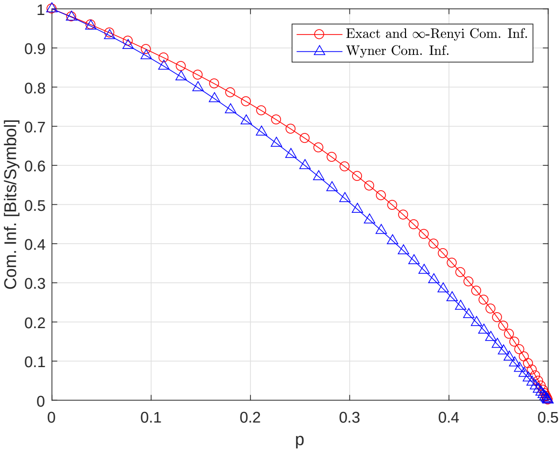
Theorem (Evaluation of Upper and Lower Bounds for DSBS(p))

For a DSBS $(X, Y) \sim \mathrm{DSBS}(p)$ *with crossover probability* $p \in (0, 1/2)$*,*

$$\begin{aligned}\tilde{T}_\infty(\pi_{XY}) &= T_{\mathrm{Ex}}(\pi_{XY}) \\ &= -2h(a) - (1-2a)\log\left[\frac{1}{2}\left(a^2 + (1-a)^2\right)\right] - 2a\log\left[a(1-a)\right],\end{aligned}$$

where $a := \frac{1-\sqrt{1-2p}}{2} \in (0, \frac{1}{2})$ *and* $h(a) := -a\log a - (1-a)\log(1-a)$*.*

Numerical Results — DSBS



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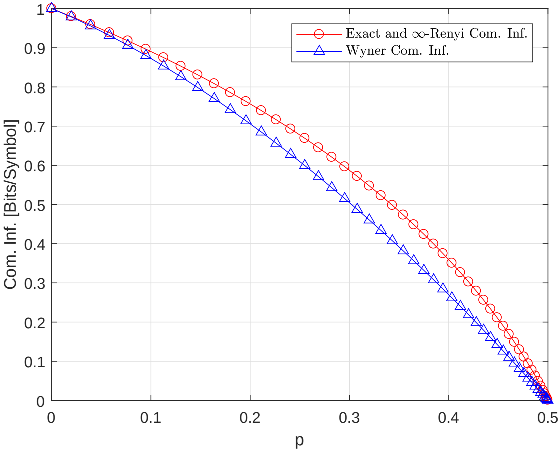

$$T_{\text{Ex}}(\text{DSBS}(p)) > C_{\text{W}}(\text{DSBS}(p)) \quad \forall\, p \in (0, 1/2).$$

Answers the open question in [Kumar et al., 2014].

Why is Exact CI (or ∞-Rényi CI) $>$ Wyner's CI?

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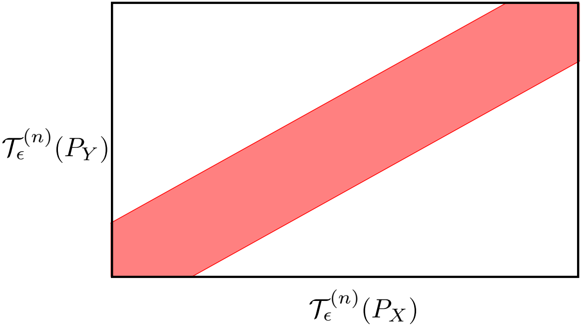

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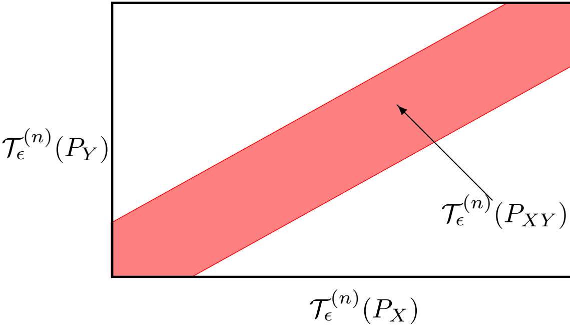
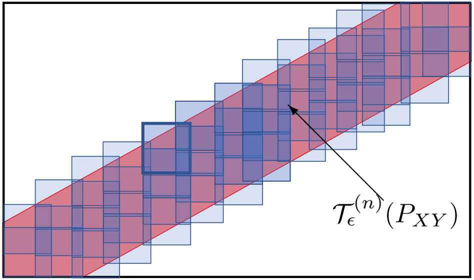


Wyner's common information requires

$$\frac{P_{X^nY^n}(x^n, y^n)}{\pi_{XY}^n(x^n, y^n)} = 1 + o(1) \quad \text{for almost all} \quad (x^n, y^n) \in \mathcal{T}_\epsilon^{(n)}(\pi_{XY})$$

Why is Exact CI (or ∞-Rényi CI) $>$ Wyner's CI?

$\mathcal{T}_\epsilon^{(n)}(P_Y)$



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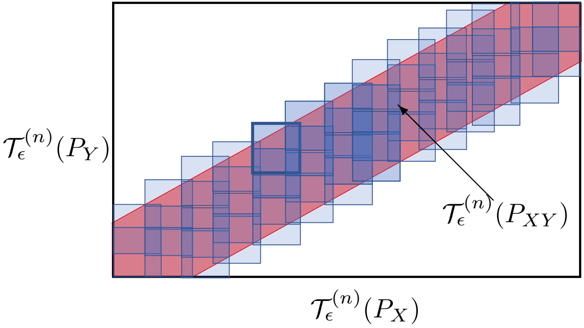

Rényi CI of order ∞ or Exact CI requires

$$\frac{P_{X^nY^n}(x^n,y^n)}{\pi^n_{XY}(x^n,y^n)} = 1 + o(1) \quad \text{for all} \quad (x^n,y^n) \in \mathrm{supp}(P_{X^nY^n})$$

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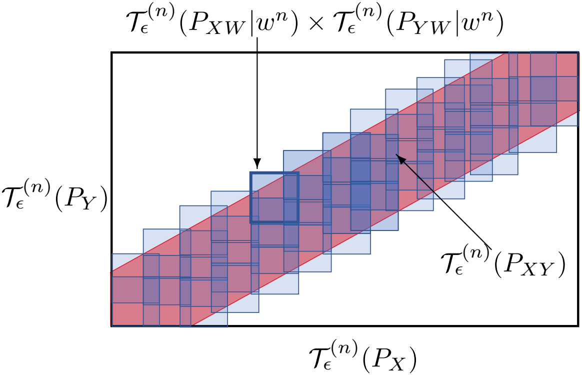

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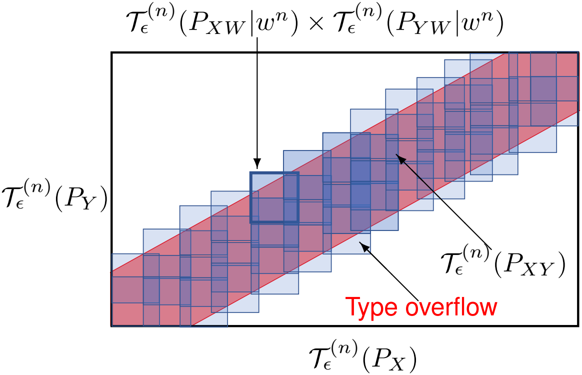

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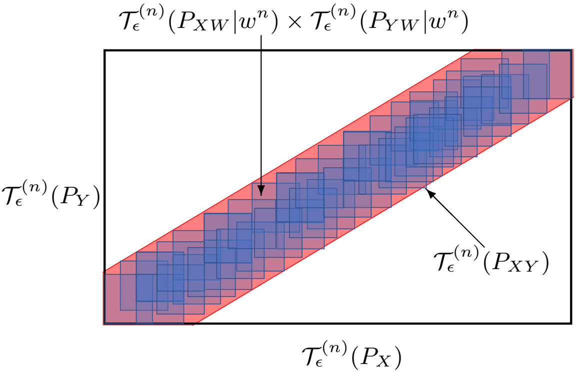

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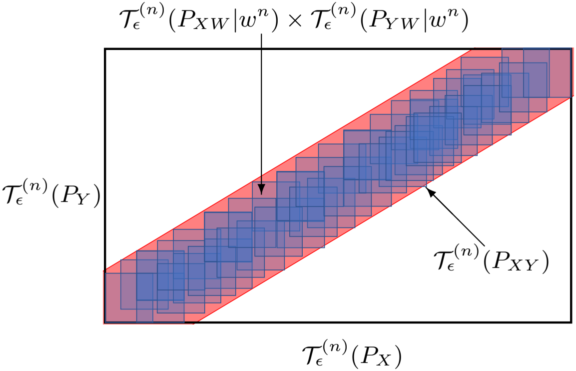

Sufficient Condition [Vellambi and Kliewer, 2016]

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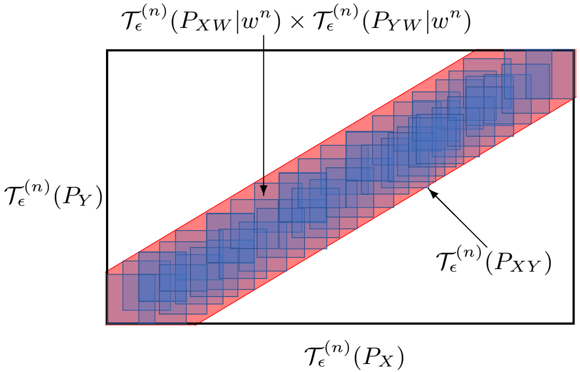

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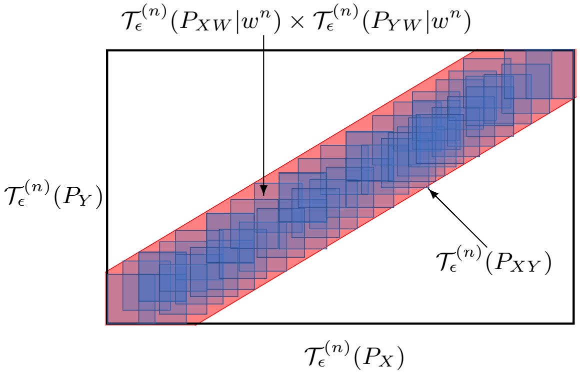

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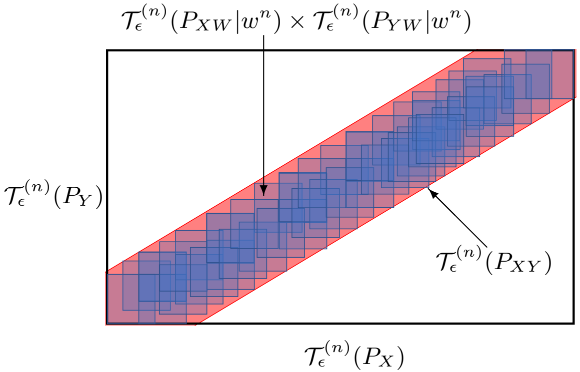

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$$\iff \mathcal{T}_\epsilon^{(n)}(P_{XY}) \approx \bigcup_{w^n\in\mathcal{C}} \Big(\mathcal{T}_\epsilon^{(n)}(P_{XW}|w^n) \times \mathcal{T}_\epsilon^{(n)}(P_{YW}|w^n)\Big)$$

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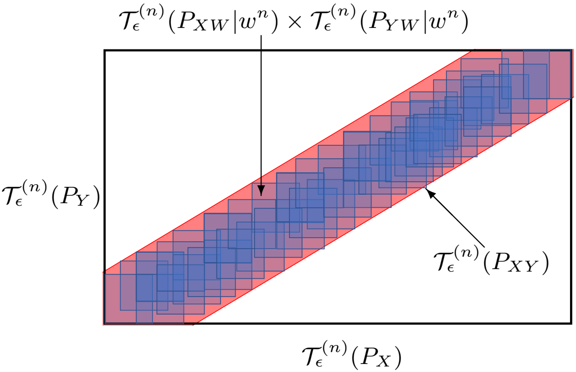

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$$\Longleftrightarrow \mathcal{T}_\epsilon^{(n)}(P_{XY}) \approx \mathrm{supp}(P_{X^nY^n}) \qquad \text{(No type overflow)}$$

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Example for Sufficient Condition:

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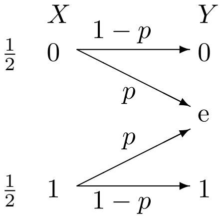

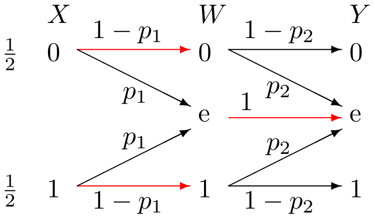

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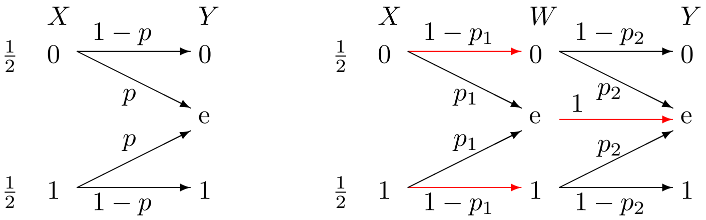

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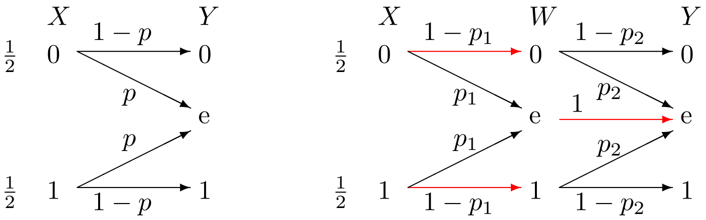


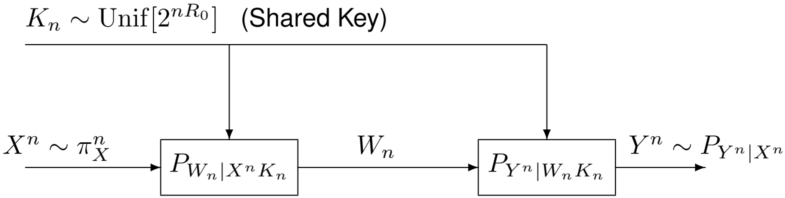
- $(1-p_1)(1-p_2)=1-p$.
- The Exact CI is equal to Wyner's CI and

$$\tilde{T}_\infty(\pi_{XY}) = T_{\text{Exact}}(\pi_{XY}) = C_{\text{Wyner}}(\pi_{XY}) = \begin{cases} 1 & p \le \frac{1}{2} \\ H(p) & p > \frac{1}{2} \end{cases}.$$

Outline

Channel Synthesis

- Given $\pi_{XY} = \pi_X \pi_{Y|X}$ consider the following task:



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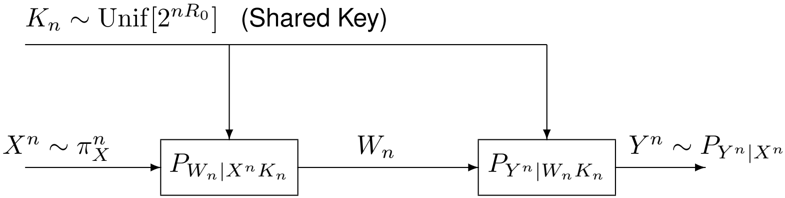

- Goal: Ensure that

$$P_{X^nY^n} \approx \pi_{XY}^n \text{ (Approximate)} \quad \text{or} \quad P_{X^nY^n} = \pi_{XY}^n \text{ (Exact)}.$$

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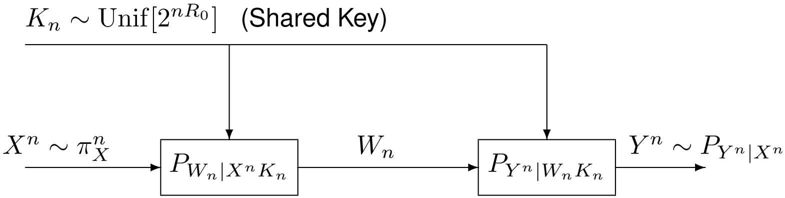

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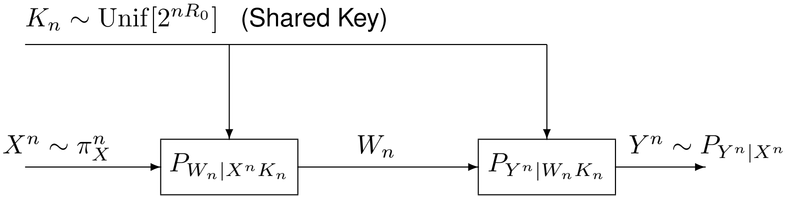

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- Known as channel synthesis [Cuff, 2012].

Approximate Channel Synthesis

- Consider approximate channel synthesis under TV criterion, i.e.,

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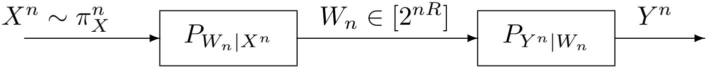

so the minimum compression rate is Wyner's common information

$$R^*(R_0 = 0|\pi_{XY}) = C_{\mathrm{W}}(\pi_{XY})$$

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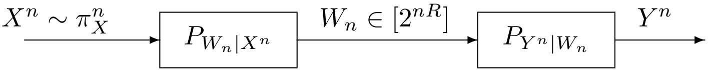

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$$R^*(R_0 = 0|\pi_{XY}) = C_{\mathrm{W}}(\pi_{XY})$$

- When $R_0 = \infty$,

$$R^*(R_0 = \infty|\pi_{XY}) = I_\pi(X;Y)$$

Approximate Channel Synthesis

It was shown in [Cuff, 2012] that

$$\mathcal{R}_{\mathrm{W}}(\pi_{XY}) := \bigcup_{\substack{P_W P_{X|W} P_{Y|W}: \\ P_{XY}=\pi_{XY}}} \left\{ (R, R_0) \, : \, \begin{array}{r} R \geq I(X;W) \\ R + R_0 \geq I(XY;W) \end{array} \right\}.$$

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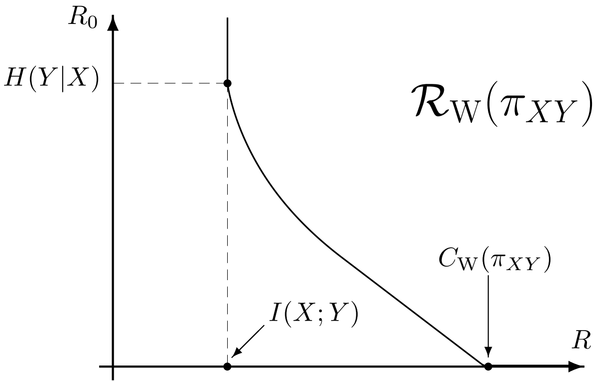
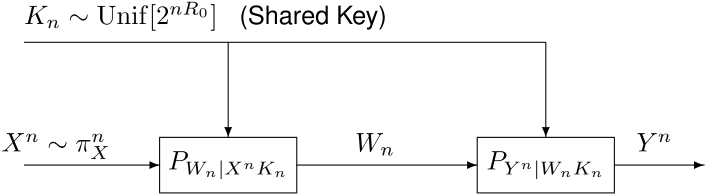


Exact Channel Synthesis



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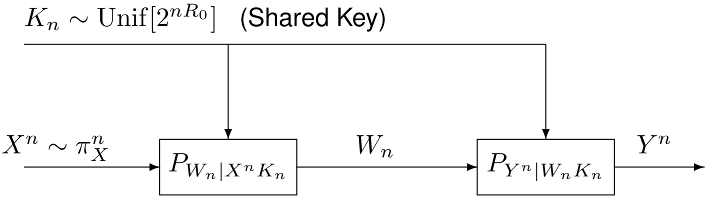
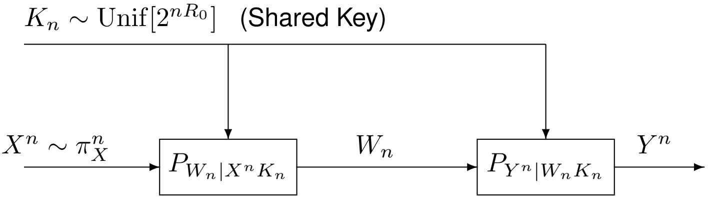


- Now, similarly to exact common information, we demand that

$$P_{X^nY^n} = \pi_{XY}^n \quad \text{for some large enough} \quad n \in \mathbb{N}$$

but just like exact CI, we allow variable-length codes for W_n.

Exact Channel Synthesis



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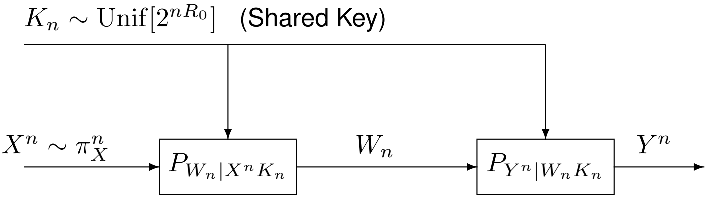

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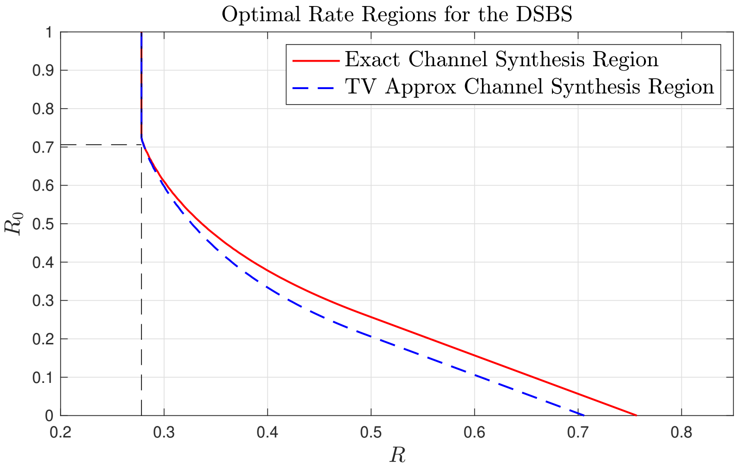
Doubly Binary Symmetric Sources



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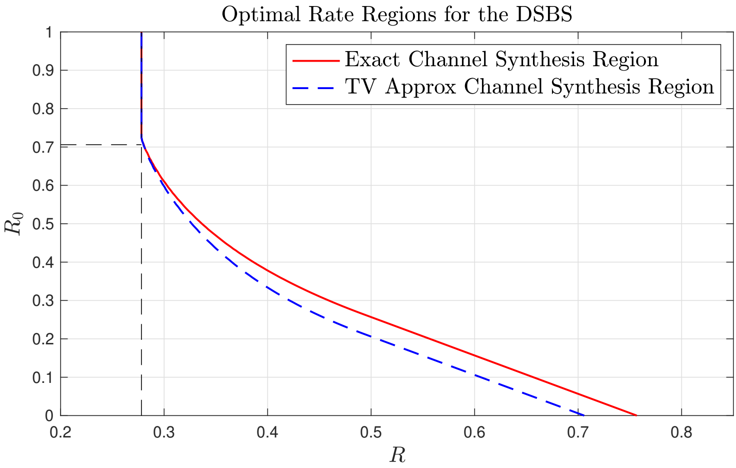

Exact channel synthesis region is strictly smaller than $\mathcal{R}_{\mathrm{W}}(\pi_{XY})$

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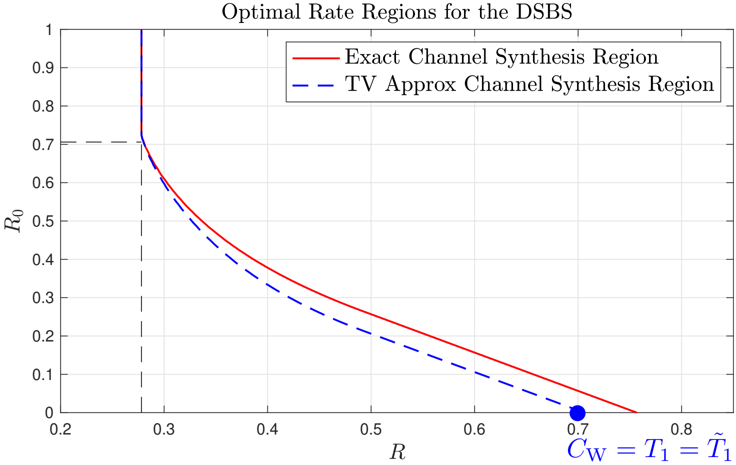

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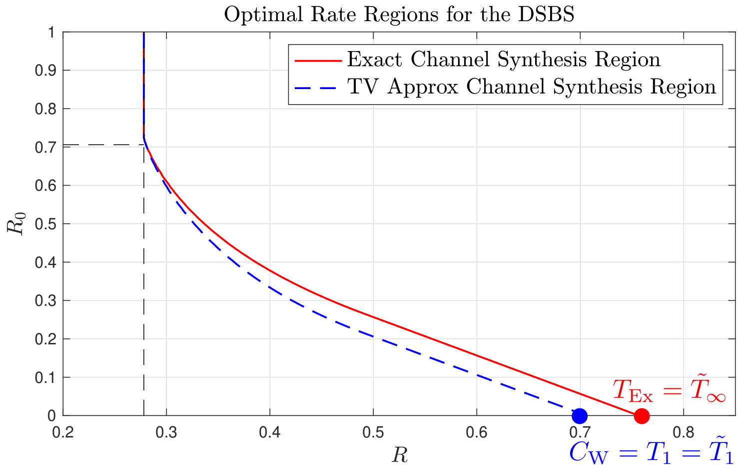

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Outline

Nonnegative Matrix Factorization

- Given a matrix $\mathbf{M} \in \mathbb{R}_+^{m \times k}$, find $\mathbf{U} \in \mathbb{R}_+^{m \times r}$ and $\mathbf{V} \in \mathbb{R}_+^{r \times k}$ such that

$$\mathbf{M} \approx \mathbf{U}\mathbf{V} \qquad \text{or} \qquad \mathbf{M} = \mathbf{U}\mathbf{V}.$$

Many applications. See [Cichocki et al., 2009] or [Gillis, 2020].

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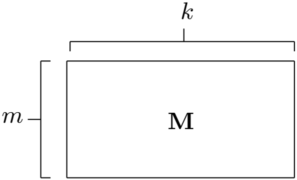

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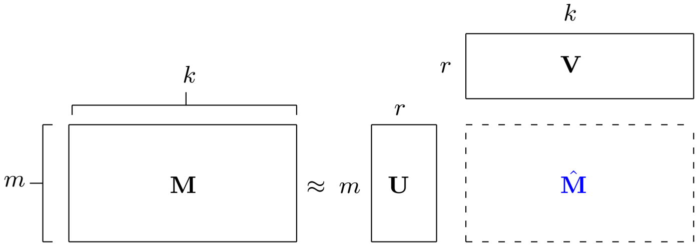

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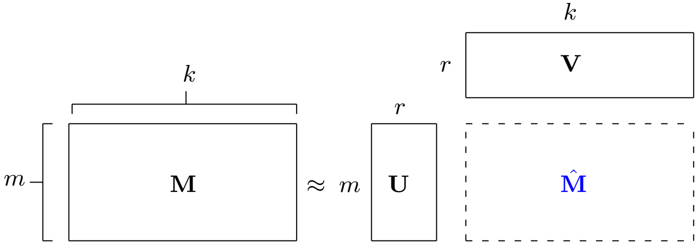

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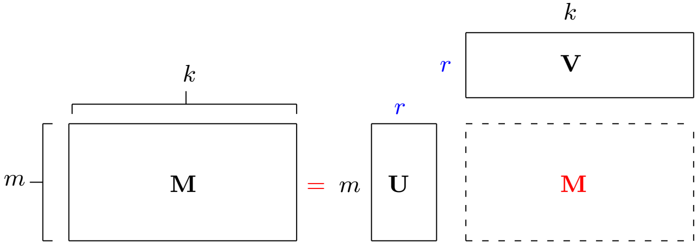

- Only interested in exact factorization.
- What is the minimum r to achieve exact factorization? Is this connected to information theory?

Nonnegative Rank

Definition

The nonnegative rank of $\mathbf{M} \in \mathbb{R}_+^{m\times k}$, denoted as $\mathrm{rank}_+(\mathbf{M})$, is the smallest integer r such that

$$\mathbf{M} = \sum_{w=1}^{r} \mathbf{u}_w \mathbf{v}_w^\top$$

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$$\mathbf{M} = \begin{bmatrix} 0 & (a_1-a_2)^2 & (a_1-a_3)^2 & \ldots & (a_1-a_m)^2 \\ (a_2-a_1)^2 & 0 & (a_2-a_3)^2 & \ldots & (a_2-a_m)^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ (a_m-a_1)^2 & (a_m-a_2)^2 & (a_m-a_3)^2 & \ldots & 0 \end{bmatrix}.$$

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- $\mathrm{rank}(\mathbf{M}) \le 3$. [Beasley and Laffey, 2009] showed $\mathrm{rank}_+(\mathbf{M}) = \Omega(\log m)$.

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- Wyner's common information for $\mathbf{M}$ is

$$C_{\mathrm{W}}(\mathbf{M}) := C_{\mathrm{W}}(\pi_{XY}).$$

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Theorem ([Jain et al., 2013], [Braun and Pokutta, 2013])

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$$P_{W|XY}(w|x,y) = \begin{cases} \dfrac{[\mathbf{u}_w]_x [\mathbf{v}_w]_y}{M_{x,y}} & M_{x,y} > 0 \\ \text{arbitrary} & M_{x,y} = 0 \end{cases}.$$

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So, $X - W - Y$ and

$$C_{\mathrm{W}}(\mathbf{M}) \le I_P(XY;W) \le H(W) \le \log |\mathcal{W}| = \log \mathrm{rank}_+(\mathbf{M}).$$

□

Gap Between $C_{\mathrm{W}}(\mathbf{M})$ and $\log \mathrm{rank}_{+}(\mathbf{M})$?

- Consider the diagonal matrix

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- Gap can be arbitrarily large.
- Is the relation between $C_{\mathrm{W}}(\mathbf{M})$ and $\log \mathrm{rank}_+(\mathbf{M})$ fundamental?

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Theorem ([Braun et al., 2017])

Let $\mathbf{M} \in \mathbb{R}_+^{m \times k}$ *be such that* $\|\mathbf{M}\|_1 = \sum_{x,y} M_{x,y} = 1$. *For any* $\epsilon, \delta > 0$, *if* $n \geq n_0(\epsilon, \delta, m, k, C_{\mathrm{W}}(\mathbf{M}))$ *is sufficiently large, there exists* $\mathbf{M}_{\epsilon,\delta,n} \in \mathbb{R}_+^{m^n \times k^n}$ *with*

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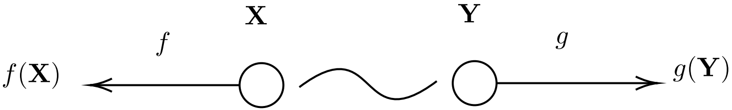

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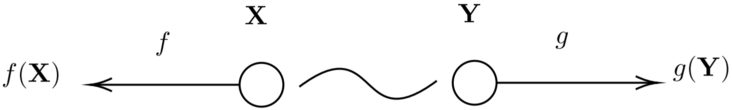
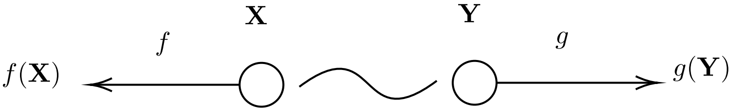


- $(\mathbf{X}, \mathbf{Y}) \sim P_{XY}^n$: a pair of correlated sources

Gács–Körner–Witsenhausen's System



- $(\mathbf{X}, \mathbf{Y}) \sim P_{XY}^n$: a pair of correlated sources
- Define one-sided ϵ-GKW common information:

$$T_X(\epsilon) := \liminf_{n\to\infty} \max_{f,g:\mathbb{P}[f(\mathbf{X})\neq g(\mathbf{Y})]\leq\epsilon} \frac{1}{n} H(f(\mathbf{X}))$$

$$T_Y(\epsilon) := \liminf_{n\to\infty} \max_{f,g:\mathbb{P}[f(\mathbf{X})\neq g(\mathbf{Y})]\leq\epsilon} \frac{1}{n} H(g(\mathbf{Y}))$$

Gács–Körner–Witsenhausen's CI

Problems of Control and Information Theory, Vol 2 (2), pp 119–162 (1973)

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P. GÁCS and J KÖRNER

(Budapest)

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$$\lim_{\epsilon \downarrow 0} T_X(\epsilon) = \lim_{\epsilon \downarrow 0} T_Y(\epsilon) = C_{\mathrm{GKW}}(X;Y),$$

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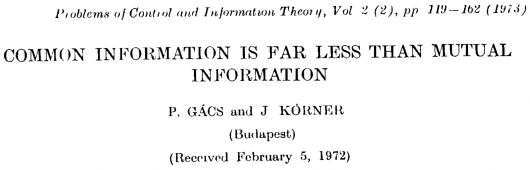
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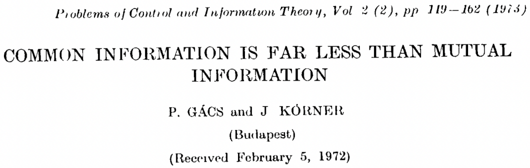
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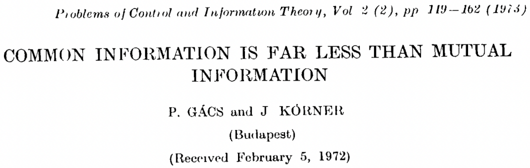
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- Abridged version of GKW's system as in [Csiszár and Narayan, 2000]
- Other interesting operational interpretations in [Yu and Tan, 2019a]

Undesirable Properties of GKW's CI

- Fact: Gács–Körner–Witsenhausen's CI $= 0$ for Gaussian sources and doubly symmetric binary sources (DSBSes)
- More unfortunately, we cannot extract even one pair of identical bits from $(\mathbf{X}, \mathbf{Y})$, if $(\mathbf{X}, \mathbf{Y})$ is jointly Gaussian or if $(\mathbf{X}, \mathbf{Y})$ is a DSBS.
- How to measure "common information" for this case?
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- A Variant of CI: What is the maximal possible **correlation of a pair of bits that can be extracted from $\mathbf{X}, \mathbf{Y}$ individually?**
- Coined the binary decision problem [Witsenhausen, 1975], the noninteractive correlation distillation (NICD) problem [Mossel et al., 2006], the noninteractive binary simulation problem [Kamath and Anantharam, 2016]

Outline

Doubly Symmetric Binary Source (DSBS)

- In this section, we only consider the DSBS

$$P_{XY} = \begin{bmatrix} \frac{1+\rho}{4} & \frac{1-\rho}{4} \\ \frac{1-\rho}{4} & \frac{1+\rho}{4} \end{bmatrix}$$

with correlation $\rho \in (0,1)$, and

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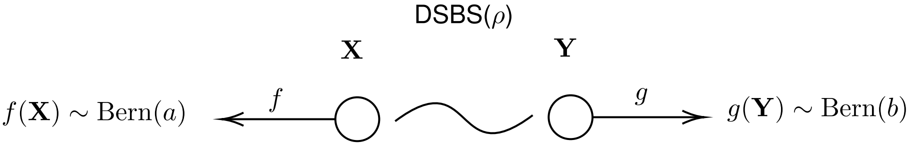
- If you are interested in other sources, please refer to [Ahlswede and Gács, 1976, Borell, 1985, Carlen and Cordero-Erausquin, 2009, Mossel and Neeman, 2015, Beigi and Nair, 2016, Yu et al., 2021, Yu, 2021b]...

Non-Interactive Correlation Distillation



$\max \mathbb{P}(f(\mathbf{X}) = g(\mathbf{Y}))$ or equivalently, $\max \mathbb{P}(f(\mathbf{X}) = g(\mathbf{Y}) = 1)$

Non-Interactive Correlation Distillation

- Formally, for $a, b \in [0,1]$, define the Forward Joint Probability as

$$\overline{\Gamma}^{(n)}(a,b) := \max_{\substack{f,g:\{0,1\}^n \to \{0,1\}: \mathbb{P}(f(\mathbf{X})=1) \le a, \\ \mathbb{P}(g(\mathbf{Y})=1) \le b}} \mathbb{P}\left(f(\mathbf{X}) = g(\mathbf{Y}) = 1\right)$$

$$= \max_{\substack{A,B \subseteq \{0,1\}^n : P_X^n(A) \le a, \\ P_Y^n(B) \le b}} P_{XY}^n(A \times B), \qquad (f = 1_A, g = 1_B)$$

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- For $a = \frac{M}{2^n}, b = \frac{N}{2^n}$ (with integers M, N), the "inequalities" in the constraints can be replaced by "equalities"
- Equivalence:

$$\overline{\Gamma}^{(\infty)}(1-a,b) = b - \underline{\Gamma}^{(\infty)}(a,b),$$

where $\overline{\Gamma}^{(\infty)}$, $\underline{\Gamma}^{(\infty)}$ denote the pointwise limits of $\overline{\Gamma}^{(n)}$, $\underline{\Gamma}^{(n)}$ as $n \to \infty$.

Asymptotic Regimes and Exponents

Asymptotic cases as $n \to \infty$

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- Central Limit (CL) regime: $a = 2^{-\alpha}, b = 2^{-\beta}$ are fixed

  (Forward and Reverse) CL Exponents: For $\alpha, \beta \in (0, \infty)$,

$$\underline{\Theta}_{\mathrm{CL}}^{(n)}(\alpha, \beta) := -\log \overline{\Gamma}^{(n)}\left(2^{-\alpha}, 2^{-\beta}\right) \qquad \overline{\Theta}_{\mathrm{CL}}^{(n)}(\alpha, \beta) := -\log \underline{\Gamma}^{(n)}\left(2^{-\alpha}, 2^{-\beta}\right)$$

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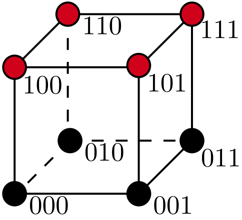
- Denote $\underline{\Theta}_{\mathrm{CL}}^{(\infty)}$, $\overline{\Theta}_{\mathrm{CL}}^{(\infty)}$, $\underline{\Theta}_{\mathrm{LD}}^{(\infty)}$, $\overline{\Theta}_{\mathrm{LD}}^{(\infty)}$, as the pointwise limits as $n \to \infty$.

Achievability: Hamming Subcubes



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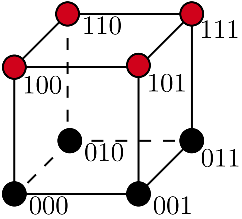

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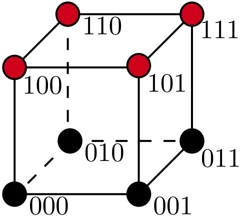

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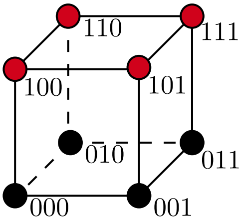

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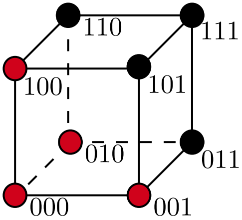
  - Special case $\mathcal{C}_{n-1}$: e.g., $\{1\} \times \{0,1\}^{n-1}$ (Indicator $\mathbf{x} \mapsto x_1$ called a dictator function)
- Case of $a = b = 2^{-k}$: $A = B = \mathcal{C}_{n-k}$ (identical) $\Longrightarrow$

$$P_{XY}^n(A \times B) = P_{XY}(1,1)^k = \left(\frac{1+\rho}{4}\right)^k$$

$A = \mathbf{1} - B = \mathcal{C}_{n-k}$ (anti-symmetric) $\Longrightarrow$

$$P_{XY}^n(A \times B) = P_{XY}(1,0)^k = \left(\frac{1-\rho}{4}\right)^k$$

Achievability: Hamming Balls (CL Regime)



- Hamming Ball: $\mathbb{B}_r(\mathbf{0}) := \{\mathbf{x} : d_{\mathrm{H}}(\mathbf{x}, \mathbf{0}) \le r\} \iff \{\mathbf{x} : \sum_{i=1}^{n} x_i \le r\}$

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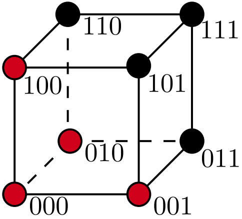

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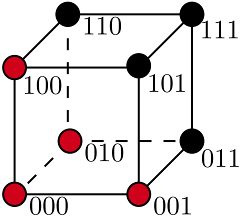

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- By the univariate and multivariate CL theorems,

$$P_X^n(A) \to \Phi(\lambda), \qquad P_Y^n(B) \to \Phi(\mu), \qquad P_{XY}^n(A \times B) \to \Phi_\rho(\lambda, \mu)$$

where Φ is the CDF of the standard Gaussian, and $\Phi_\rho(\cdot,\cdot)$ is the CDF of the zero-mean bivariate Gaussian with covariance matrix $\begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}$.

Achievability: Hamming Balls (CL Regime)

- Achievable CL probabilities:

$$\overline{\Gamma}^{(\infty)}(a,b) \geq \Lambda_{\rho}(a,b) \qquad \text{(by concentric balls)}$$

  - Bivariate normal copula (or Gaussian quadrant probability function):

$$\Lambda_{\rho}(a,b) := \Phi_{\rho}\left(\Phi^{-1}(a), \Phi^{-1}(b)\right)$$

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- By equivalence of forward and reverse joint probabilities,

$$\underline{\Gamma}^{(\infty)}(a,b) \le \Lambda_{-\rho}(a,b) \qquad \text{(by anti-concentric balls)}$$

  - $\Lambda_{-\rho}(a,b)$ is attained by anti-concentric balls $A = \mathbb{B}_{r_n}(\mathbf{0}), B = \mathbb{B}_{s_n}(\mathbf{1})$

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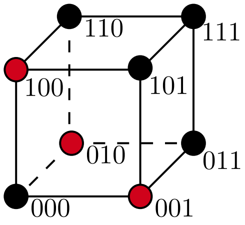
  - $\Lambda_{-\rho}(a,b)$ is attained by anti-concentric balls $A = \mathbb{B}_{r_n}(\mathbf{0}), B = \mathbb{B}_{s_n}(\mathbf{1})$
- Considering exponents,

$$\underline{\Theta}_{\mathrm{CL}}^{(\infty)}(\alpha,\beta) \leq \underline{\Theta}_{\mathrm{CL}}(\alpha,\beta) \qquad \overline{\Theta}_{\mathrm{CL}}^{(\infty)}(\alpha,\beta) \geq \overline{\Theta}_{\mathrm{CL}}(\alpha,\beta).$$

  - Exponents of Λ_{ρ} and $\Lambda_{-\rho}$:

$$\underline{\Theta}_{\mathrm{CL}}(\alpha,\beta) := -\log \Lambda_{\rho}\left(e^{-\alpha}, e^{-\beta}\right), \qquad \overline{\Theta}_{\mathrm{CL}}(\alpha,\beta) := -\log \Lambda_{-\rho}\left(e^{-\alpha}, e^{-\beta}\right)$$

Achievability: Hamming Spheres (LD Regime)



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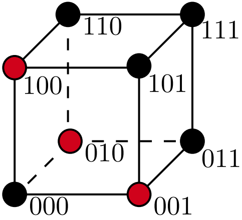

- Hamming Sphere: For $r \in [0:n]$, $\mathbb{S}_r(\mathbf{0}) := \{\mathbf{x} : d_{\mathrm{H}}(\mathbf{x}, \mathbf{0}) = r\} \Longleftrightarrow \{\mathbf{x} : \sum_{i=1}^{n} x_i = r\}$

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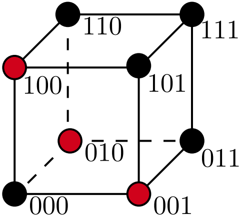

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- It can be regarded as a type class with type $(\lambda, \bar{\lambda})$ in Hamming space, where $\lambda := \frac{r}{n}$ and $\bar{\lambda} := 1 - \lambda$

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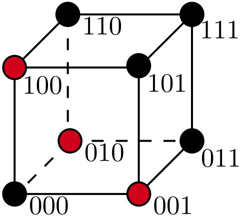

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- It can be regarded as a type class with type $(\lambda, \bar{\lambda})$ in Hamming space, where $\lambda := \frac{r}{n}$ and $\bar{\lambda} := 1 - \lambda$
- LD regime: Choose $A = \mathbb{S}_{r_n}(\mathbf{0}), B = \mathbb{S}_{s_n}(\mathbf{0})$ with $r_n = \lambda n, s_n = \mu n$ where $\lambda, \mu \in [0, 1]$

Achievability: Hamming Spheres (LD Regime)

By LD theory (or Sanov's theorem),

$$-\frac{1}{n}\log P_X^n(A) \to D\left((\lambda,\bar{\lambda})\,\|P_X\right) = 1 - H_2(\lambda)$$
$$-\frac{1}{n}\log P_Y^n(B) \to D\left((\mu,\bar{\mu})\,\|P_Y\right) = 1 - H_2(\mu)$$
$$-\frac{1}{n}\log P_{XY}^n(A\times B) \to \mathbb{D}\left((\lambda,\bar{\lambda}),(\mu,\bar{\mu})\,\|P_{XY}\right),$$

where the minimum-relative-entropy over couplings of (Q_X, Q_Y) is

$$\mathbb{D}(Q_X, Q_Y\|P_{XY}) := \min_{Q_{XY}\in\mathcal{C}(Q_X,Q_Y)} D(Q_{XY}\|P_{XY})$$

with $\mathcal{C}(Q_X, Q_Y) := \{Q_{XY} \text{ with marginals } Q_X, Q_Y\}$ denoting the coupling set of Q_X and Q_Y.

Achievability: Hamming Spheres (LD Regime)

[Ordentlich et al., 2020] proved...

- Optimizing $\mathbb{D}(Q_X, Q_Y \| P_{XY})$ over feasible $Q_X := \left(\lambda, \bar{\lambda}\right), Q_Y := (\mu, \bar{\mu}) \Longrightarrow$

$$\underline{\Theta}_{\mathrm{LD}}^{(\infty)}(\alpha, \beta) \le \underline{\Theta}_{\mathrm{LD}}(\alpha, \beta) := \min_{\substack{Q_X, Q_Y : D(Q_X \| P_X) \ge \alpha, \\ D(Q_Y \| P_Y) \ge \beta}} \mathbb{D}(Q_X, Q_Y \| P_{XY}),$$

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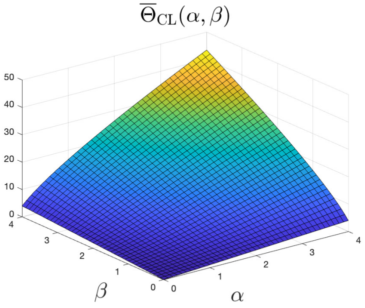
[Ordentlich et al., 2020] conjectured...

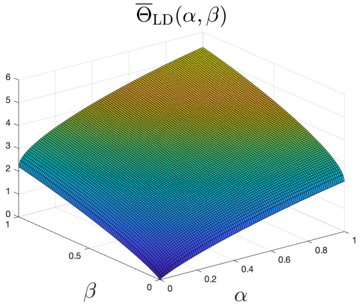
Conjecture (Ordentlich–Polyanskiy–Shayevitz (2020))

For the DSBS and $\alpha, \beta \in (0, 1)$,

$$\underline{\Theta}_{\mathrm{LD}}^{(\infty)}(\alpha, \beta) \stackrel{?}{=} \underline{\Theta}_{\mathrm{LD}}(\alpha, \beta), \qquad \overline{\Theta}_{\mathrm{LD}}^{(\infty)}(\alpha, \beta) \stackrel{?}{=} \overline{\Theta}_{\mathrm{LD}}(\alpha, \beta).$$

Exponents induced by Hamming Spheres for $\rho = 0.9$





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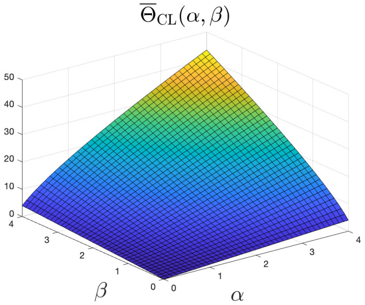

Remark that $\overline{\Theta}_{\mathrm{LD}}$ looks concave! Has implications for OPS' conjecture.

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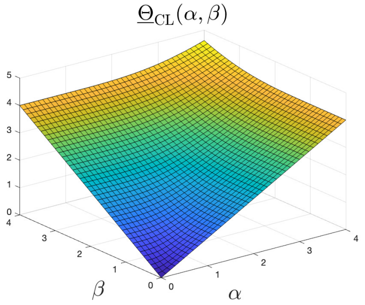

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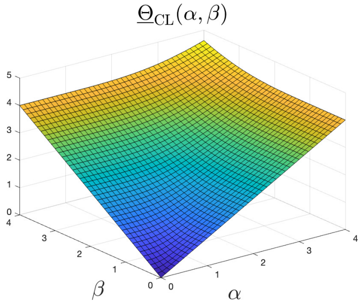

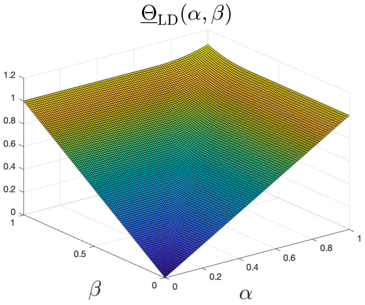

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Comparison: Hamming Subcubes vs. Hamming Balls

Regime	Central Limit		Large Deviation
a, b	fixed and large a, b	fixed but small a, b	exp. small a, b
Subcubes	Better	Worse	Worse
Balls	Worse	Better	Better

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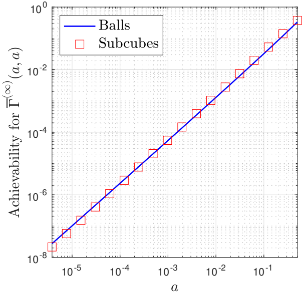
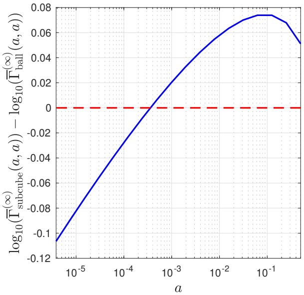
- For large a, b, subcubes are better; for small a, b, balls are better





Natural Questions on Optimality I

- Question: Are Hamming subcubes optimal for large a, b (CL regime)?
- Are subcubes optimal for $a = b \in \left\{\frac{1}{2}, \frac{1}{4}\right\}$?
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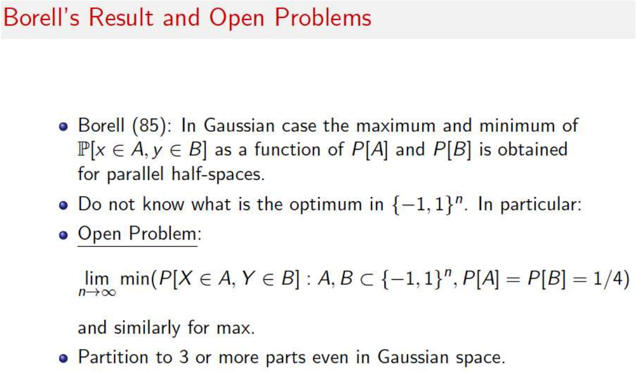

Natural Questions on Optimality II

- Question: Are Hamming balls optimal for exp. small a, b (LD regime)?
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Our interest is in the greatest and smallest exponential decay rate of $P_{XY}(A \times B)$ among all possible sets A, B of sizes $2^{n\alpha}$ and $2^{n\beta}$, respectively. To that end, for fixed $0 < \alpha, \beta < 1$ we define

$$\overline{E}(\alpha, \beta, \rho) \triangleq -\limsup_{n\to\infty} \max_{\{A\},\{B\}} \frac{1}{n} \log P_{XY}(A \times B), \quad (8)$$

$$\underline{E}(\alpha, \beta, \rho) \triangleq -\liminf_{n\to\infty} \min_{\{A\},\{B\}} \frac{1}{n} \log P_{XY}(A \times B), \quad (9)$$

where $\max_{\{A\},\{B\}}$ and $\min_{\{A\},\{B\}}$ denote optimizations over the sequences of sets $A_n \subset \{0,1\}^n$, $B_n \subset \{0,1\}^n$, $n \in \mathbb{Z}_+$ such that

$$|A_n| = 2^{n\alpha + o(n)}, \qquad |B_n| = 2^{n\beta + o(n)}.$$

Our **main conjecture** is that *both* $\overline{E}(\alpha, \beta, \rho)$ *and* $\underline{E}(\alpha, \beta, \rho)$ *are optimized by concentric (resp., anti-concentric) Hamming balls*. In this work we show partial progress towards establishing this conjecture. Our conjecture is in line with the well-known facts that among all pairs of sets $A, B \subset \{0,1\}^n$ of given sizes, the maximal distance $d_{\max}(A, B) = \max_{a\in A, b\in B} d(a, b)$ is minimized by concentric Hamming (quasi) balls [19], [20], whereas the minimum distance $d_{\min}(A, B) = \min_{a\in A, b\in B} d(a, b)$ is maximized by anti-concentric Hamming (quasi) balls [21].



Converse for $a = b = \frac{1}{2}$: Subcubes/dictators optimal?

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$$\rho_{\mathrm{m}}(X;Y) := \sup_{f,g} \rho(f(X);g(Y)),$$

  - $\rho(U;V) := \frac{\mathbb{E}[UV]}{\sqrt{\mathrm{var}[U]\mathrm{var}[V]}}$ is the Pearson correlation coefficient
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- For binary X, Y, $\rho_{\mathrm{m}}(X;Y) = |\rho(X;Y)|$.

Converse for $a = b = \frac{1}{2}$: Subcubes/dictators optimal?

Theorem ([Witsenhausen, 1975])

Let $\bar{a} = 1 - a$*. For any* A, B *with* $P_X^n(A) = a, P_Y^n(B) = b$*,*

$$ab - \rho\sqrt{a\bar{a}b\bar{b}} \le P_{XY}^n(A \times B) \le ab + \rho\sqrt{a\bar{a}b\bar{b}}.$$

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Proof: Setting $U = 1_A(\mathbf{X}), V = 1_B(\mathbf{Y})$, we have $U - \mathbf{X} - \mathbf{Y} - V$

$$\frac{|P_{XY}^n(A \times B) - ab|}{\sqrt{a\bar{a}}\sqrt{b\bar{b}}} = |\rho(U;V)|$$

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$$= \rho_{\mathrm{m}}(U;V) \qquad \text{[Binary]}$$

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&= \rho.
\end{aligned}$$

Converse for $a = b = \frac{1}{2}$: Subcubes/dictators optimal?

Important Consequence:

- For $a = b = 1/2$,

$$\frac{1-\rho}{4} \le P_{XY}^n(A \times B) \le \frac{1+\rho}{4}.$$

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- Dictators (subcubes) are optimal for $a = b = 1/2$, i.e.,

$$\overline{\Gamma}^{(n)}\left(\frac{1}{2}, \frac{1}{2}\right) = \frac{1+\rho}{4} \qquad \underline{\Gamma}^{(n)}\left(\frac{1}{2}, \frac{1}{2}\right) = \frac{1-\rho}{4}$$

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- Define the k-degree Fourier weight as

$$\mathbf{W}_k[f] := \sum_{|\mathbf{y}|=k} \hat{f}(\mathbf{y})^2$$

where $|\mathbf{y}|$ denotes the Hamming weight of $\mathbf{y}$.

Converse for $a = b = \frac{1}{4}$: Are subcubes optimal?

- Properties: For a Boolean f with mean a,

$$\mathbf{W}_0[f] = a^2 \qquad \sum_{k=0}^{n} \mathbf{W}_k[f] = a$$

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- Linear Programming bound on $\mathbf{W}_1[f]$ [Fu et al., 2001, Yu and Tan, 2019b]:

$$\mathbf{W}_1[f] \le \varphi(a) := \begin{cases} 2a\left(\sqrt{a} - a\right) & 0 \le a \le 1/4 \\ a/2 & 1/4 < a \le 1/2 \end{cases}$$

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- Fact (Cauchy–Schwarz inequality):

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Suffices to consider identical Boolean functions for $\overline{\Gamma}^{(n)}(a, a)$.

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Theorem ([Yu and Tan, 2021])

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- However, $\underline{\Gamma}^{(n)}\left(\frac{1}{4},\frac{1}{4}\right)$ is still open!

Converse for LD: Strong Small-Set Expansion Theorem

Theorem (Strong Small-Set Expansion [Yu et al., 2021, Yu, 2021b])

For any $n \geq 1$ and $\alpha, \beta \in (0, 1]$,

$$\underline{\Theta}_{\mathrm{LD}}^{(n)}(\alpha, \beta) \geq \mathbb{L}\left[\underline{\Theta}_{\mathrm{LD}}\right](\alpha, \beta) \quad \text{and}$$
$$\overline{\Theta}_{\mathrm{LD}}^{(n)}(\alpha, \beta) \leq \mathbb{U}\left[\overline{\Theta}_{\mathrm{LD}}\right](\alpha, \beta),$$

where $\mathbb{L}[f]$ and $\mathbb{U}[f]$ respectively denote the lower convex and upper concave envelopes of a function f.

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Theorem (Strong Small-Set Expansion [Yu et al., 2021, Yu, 2021b])

For any $n \geq 1$ and $\alpha, \beta \in (0,1]$,

$$\underline{\Theta}_{\mathrm{LD}}^{(n)}(\alpha,\beta) \geq \mathbb{L}\left[\underline{\Theta}_{\mathrm{LD}}\right](\alpha,\beta) \quad \text{and}$$
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- Recall: $\underline{\Theta}_{\mathrm{LD}}(\alpha,\beta), \overline{\Theta}_{\mathrm{LD}}(\alpha,\beta)$ are achieved by spheres/balls
- Consequence: Time-sharing certain Hamming spheres/balls is optimal in LD regime! —— A weaker version of OPS's conjecture

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Lemma ([Yu, 2021a])

$\underline{\Theta}_{\mathrm{LD}}$ *is convex, and* $\overline{\Theta}_{\mathrm{LD}}$ *is concave.*

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- Note:
  - The limiting cases as $\rho \to 0$ or 1 were previously proven in [Ordentlich et al., 2020].
  - The special case with $\alpha = \beta$ was previously proven in [Kirshner and Samorodnitsky, 2021].

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