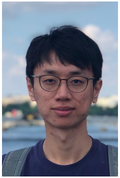
Community Detection & Matrix Completion with Two-Sided Graph Side-Information

BITS 2020

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Outline

1. Recommender Systems
2. Model
3. Main results
4. Examples
5. Proof Sketches
6. Efficient algorithm
7. Conclusion

Recommender Systems

NETFLIX YouTube TikTok

Recommender Systems

NETFLIX YouTube TikTok

	Movie 1	Movie 2	Movie 3	Movie 4	Movie 5
User 1	1	?	?	0	?
User 2	?	?	1	?	?
User 3	?	0	?	?	?
User 4	0	?	?	1	?
User 5	?	?	0	?	?

Ratings: 1 (like), 0 (dislike),
? (unknowns)

Recommender Systems

1	?	?	0	?
?	?	1	?	?
?	0	?	?	?
0	?	?	1	?
?	?	0	?	?

Ratings: 1 (like), 0 (dislike), ? (unknowns)

Goal: infer unknowns

NETFLIX YouTube TikTok

1	?	?	0	?
?	?	1	?	?
?	0	?	?	?
0	?	?	1	?
?	?	0	?	?

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Challenges: few ratings available

NETFLIX YouTube TikTok

1	?	?	0	?
?	?	1	?	?
?	0	?	?	?
0	?	?	1	?
?	?	0	?	?
?	?	?	?	?

Ratings: 1 (like), 0 (dislike), ? (unknowns)

Goal: infer unknowns

Challenges: few ratings available
cold start problem

Graph side-information helps!

1. User-similarity graph
   - Many algorithmic analyses: [Tang et al. ’13] (survey)
   - Not many theoretical analyses: [Ahn et al. ’18]

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2. Item-similarity graph

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?	?	0	?	?
?	?	?	?	?

1. User-similarity graph
   - Many algorithmic analyses: [Tang et al. ’13] (survey)
   - Not many theoretical analyses: [Ahn et al. ’18]
2. Item-similarity graph
3. Benefits of two graphs?

Model

- n users: Men ($\mathcal{M}$), Women ($\mathcal{W}$)

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	Action	Romance
Men	1	0
Women	0	1

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- *Atypical* action movies $\mathcal{A}_0$ and romance movies $\mathcal{R}_0$
- $|\mathcal{A}_0|$, $|\mathcal{R}_0|$ unknown

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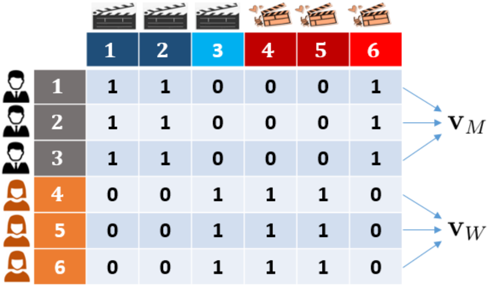
	Action		Romance	
	Typical	Atypical	Typical	Atypical
Men	1	0	0	1
Women	0	1	1	0

Non-personalized binary rating matrix

$$B = \{B_{ij} : i \in [n], j \in [m]\}$$

Personalized binary rating matrix V, where

$$V_{ij} = \begin{cases} B_{ij} \oplus \mathrm{Bern}(\theta_{\mathcal{A}}), & \text{if } j \in \mathcal{A}; \\ B_{ij} \oplus \mathrm{Bern}(\theta_{\mathcal{R}}), & \text{if } j \in \mathcal{R}. \end{cases}$$



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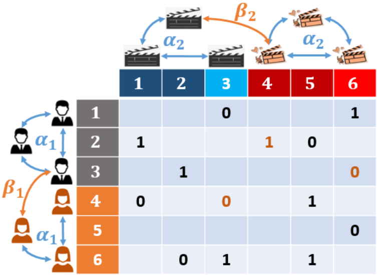
	1	2	3	4	5	6
1	1	0	0	0	0	1
2	1	1	0	1	0	1
3	1	1	0	0	0	0
4	0	0	0	1	1	0
5	1	0	1	1	1	0
6	0	0	1	0	1	0

1. Partially observed personalized binary rating matrix V^{Ω}
   - Sample probability: p



	1	2	3	4	5	6
1			0			1
2	1			1	0	
3		1				0
4	0		0		1	
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Model (Observations I)

1. Partially observed personalized binary rating matrix V^{Ω}
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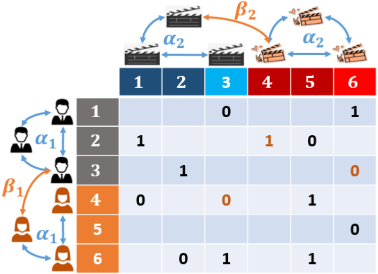
2. Row graph G_1 (users)
   - Stochastic block model (SBM)
   - Intra-prob. (α_1), inter-prob. (β_1)

$$\alpha_1, \beta_1 = \Theta\left(\frac{\log n}{n}\right),$$



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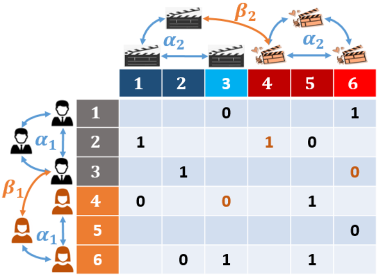

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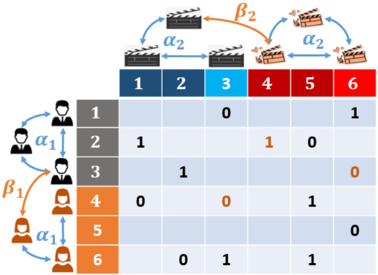

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3. Column graph G_2 (movies)
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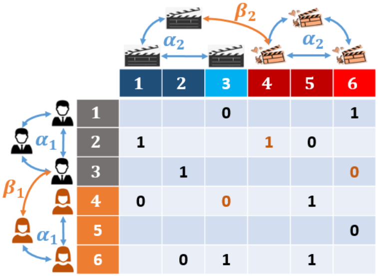

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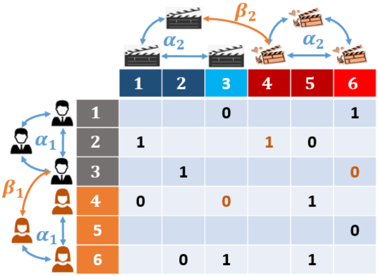

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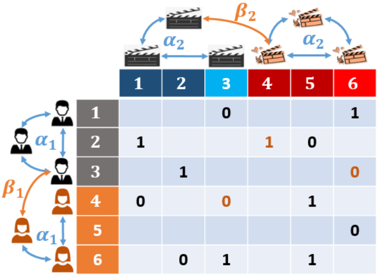

Model (Observations II)

1. Partially observed personalized binary rating matrix V^{Ω}
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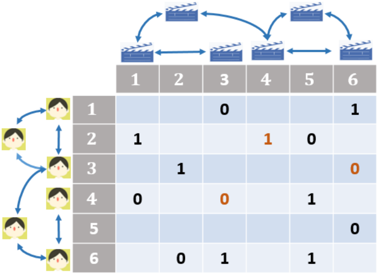

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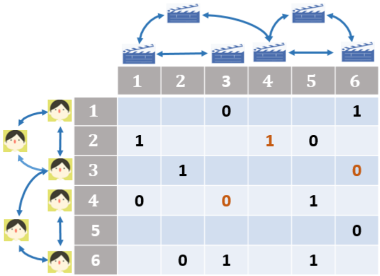
Need to cluster!



	1	2	3	4	5	6
1			0			1
2	1			1	0	
3		1				0
4	0		0		1	
5						0
6		0	1		1	

Model (task)

- Goal: recover $\mathcal{M}$, $\mathcal{W}$, $\mathcal{A}$, $\mathcal{R}$, $\mathcal{A}_0$, $\mathcal{R}_0$ based on V^{Ω}, G_1, G_2

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- $\Xi \triangleq \{\text{valid } \xi_{\mathcal{M},\mathcal{W},\mathcal{A},\mathcal{R},\mathcal{A}_0,\mathcal{R}_0}\}$.

Definition (Exact recovery)

For any estimator ϕ, the max prob. of error is

$$P_{\text{err}}(\phi) \triangleq \max_{\xi \in \Xi} \mathbb{P}_\xi(\phi(V^\Omega, G_1, G_2) \neq \xi),$$

A sequence of estimators $\phi = \{\phi_n\}_{n=1}^\infty$ satisfies the exact recovery property if

$$\lim_{n \to \infty} P_{\text{err}}(\phi_n) = 0.$$

Model (task)

- Recover $\xi_{\mathcal{M},\mathcal{W},\mathcal{A},\mathcal{R},\mathcal{A}_0,\mathcal{R}_0} \Rightarrow$ Recover non-personalized matrix B

	1	2	3	4	5	6
1	1	1	0	0	0	1
2	1	1	0	0	0	1
3	1	1	0	0	0	1
4	0	0	1	1	1	0
5	0	0	1	1	1	0
6	0	0	1	1	1	0

Model (task)

- Recover $\xi_{\mathcal{M},\mathcal{W},\mathcal{A},\mathcal{R},\mathcal{A}_0,\mathcal{R}_0} \Rightarrow$ Recover non-personalized matrix B
- Recover non-personalized matrix $B \nRightarrow$ Recover $\xi_{\mathcal{M},\mathcal{W},\mathcal{A},\mathcal{R},\mathcal{A}_0,\mathcal{R}_0}$

	1	2	3	4	5	6
1	1	1	0	0	0	1
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Model (task)

- Recover $\xi_{\mathcal{M},\mathcal{W},\mathcal{A},\mathcal{R},\mathcal{A}_0,\mathcal{R}_0} \Rightarrow$ Recover non-personalized matrix B
- Recover non-personalized matrix $B \not\Rightarrow$ Recover $\xi_{\mathcal{M},\mathcal{W},\mathcal{A},\mathcal{R},\mathcal{A}_0,\mathcal{R}_0}$

	1	2	3	4	5	6
1	1	1	0	0	0	1
2	1	1	0	0	0	1
3	1	1	0	0	0	1
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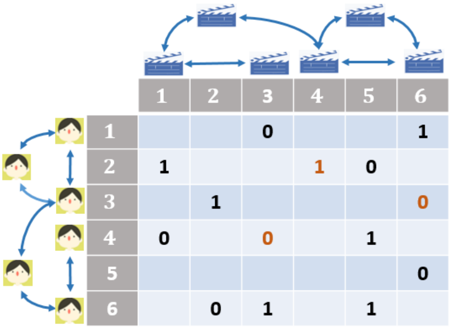
6	0	0	1	1	1	0

- Our task of learning $\mathcal{M}$, $\mathcal{W}$, $\mathcal{A}$, $\mathcal{R}$, $\mathcal{A}_0$, and $\mathcal{R}_0$ is strictly more difficult than only recovering the binary rating matrix B.

Example (Question)

What is the **minimum sample complexity** (nmp) for exact recovery?



	1	2	3	4	5	6
1			0			1
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4	0		0		1	
5						0
6		0	1		1	

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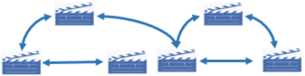

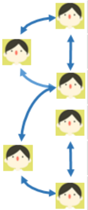

	1	2	3	4	5	6
1			0			1
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4	0		0		1	
5						0
6		0	1		1	

$$nmp = \mathbb{E}\left[\text{number of available entries in observation matrix } V^{\Omega}\right].$$

Main results

Theorem (Achievability)

(a) For $\theta_{\mathcal{A}} \neq \theta_{\mathcal{R}}$*, if*

$$p > \max\left\{ \frac{(2 - I_1)\log n}{(\nu_{\mathcal{A}\mathcal{A}} + \nu_{\mathcal{R}\mathcal{R}})m}, \frac{\log m}{\min\{\nu_{\mathcal{A}\mathcal{A}}, \nu_{\mathcal{R}\mathcal{R}}\} \cdot n}, \frac{(2 - I_2)\log m}{2\tau_{\mathcal{A}\mathcal{R}} n} \right\},$$

(b) for $\theta_{\mathcal{A}} = \theta_{\mathcal{R}}$*, if* $I_2 > 2$ *and*

$$p > \max\left\{ \frac{(2 - I_1)\log n}{(\nu_{\mathcal{A}\mathcal{A}} + \nu_{\mathcal{R}\mathcal{R}})m}, \frac{\log m}{\min\{\nu_{\mathcal{A}\mathcal{A}}, \nu_{\mathcal{R}\mathcal{R}}\} \cdot n} \right\},$$

then there exists an estimator ϕ *satisfying* $\lim_{n\to\infty} P_{\mathrm{err}}(\phi) = 0$.

- I_1 and I_2 are "qualities" of G_1 and G_2
- $\nu_{\mathcal{A}\mathcal{A}}, \nu_{\mathcal{R}\mathcal{R}}$: functions of $\theta_{\mathcal{A}}$ and $\theta_{\mathcal{R}}$.

Theorem (Converse)

(a) For $\theta_{\mathcal{A}} \neq \theta_{\mathcal{R}}$, if

$$p < \max\left\{ \frac{(2 - I_1)\log n}{(\nu_{\mathcal{A}\mathcal{A}} + \nu_{\mathcal{R}\mathcal{R}})m}, \frac{\log m}{\min\{\nu_{\mathcal{A}\mathcal{A}}, \nu_{\mathcal{R}\mathcal{R}}\} \cdot n}, \frac{(1 - I_2)\log m}{2\tau_{\mathcal{A}\mathcal{R}} n} \right\},$$

(b) for $\theta_{\mathcal{A}} = \theta_{\mathcal{R}}$, if $I_2 < 2$ or

$$p < \max\left\{ \frac{(2 - I_1)\log n}{(\nu_{\mathcal{A}\mathcal{A}} + \nu_{\mathcal{R}\mathcal{R}})m}, \frac{\log m}{\min\{\nu_{\mathcal{A}\mathcal{A}}, \nu_{\mathcal{R}\mathcal{R}}\} \cdot n} \right\},$$

then $\lim_{n\to\infty} P_{\mathrm{err}}(\phi) = 1$ for any estimator ϕ (i.e., a strong converse).

- Bounds match for $\theta_{\mathcal{A}} = \theta_{\mathcal{R}}$

$$p > \max\left\{\frac{(2 - I_1)\log n}{(\nu_{\mathcal{A}\mathcal{A}} + \nu_{\mathcal{R}\mathcal{R}})m}, \frac{\log m}{\min\{\nu_{\mathcal{A}\mathcal{A}}, \nu_{\mathcal{R}\mathcal{R}}\} \cdot n}\right\}$$

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Main results

- Bounds match for $\theta_{\mathcal{A}} = \theta_{\mathcal{R}}$

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- Bounds match up to a constant factor of 2 for $\theta_{\mathcal{A}} \neq \theta_{\mathcal{R}}$,

$$p > \max\left\{\frac{(2-I_1)\log n}{(\nu_{\mathcal{A}\mathcal{A}}+\nu_{\mathcal{R}\mathcal{R}})m}, \frac{\log m}{\min\{\nu_{\mathcal{A}\mathcal{A}}, \nu_{\mathcal{R}\mathcal{R}}\}\cdot n}, \frac{(2-I_2)\log m}{2\tau_{\mathcal{A}\mathcal{R}}n}\right\}$$
$$p < \max\left\{\frac{(2-I_1)\log n}{(\nu_{\mathcal{A}\mathcal{A}}+\nu_{\mathcal{R}\mathcal{R}})m}, \frac{\log m}{\min\{\nu_{\mathcal{A}\mathcal{A}}, \nu_{\mathcal{R}\mathcal{R}}\}\cdot n}, \frac{(1-I_2)\log m}{2\tau_{\mathcal{A}\mathcal{R}}n}\right\}$$

- Achievability:

$$p > \frac{\log n}{n} \cdot \max\left\{ \underbrace{\frac{(2 - I_1)}{(\nu_{\mathcal{A}\mathcal{A}} + \nu_{\mathcal{R}\mathcal{R}})}}_{T_1}, \underbrace{\frac{1}{\min\{\nu_{\mathcal{A}\mathcal{A}}, \nu_{\mathcal{R}\mathcal{R}}\}}}_{T_2}, \underbrace{\frac{2 - I_2}{2\tau_{\mathcal{A}\mathcal{R}}}}_{T_3} \right\},$$

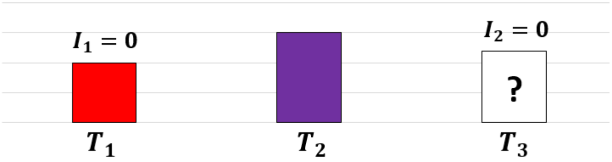
because $\min\{\nu_{\mathcal{A}\mathcal{A}}, \nu_{\mathcal{R}\mathcal{R}}\} \leq \frac{\nu_{\mathcal{A}\mathcal{A}} + \nu_{\mathcal{R}\mathcal{R}}}{2} \leq \frac{\nu_{\mathcal{A}\mathcal{A}} + \nu_{\mathcal{R}\mathcal{R}}}{2 - I_1}$.

Example 1 when $n = m$ (also includes $n \leq m$)

- Achievability:

$$p > \frac{\log n}{n} \cdot \max\left\{ \underbrace{\frac{(2 - I_1)}{(\nu_{\mathcal{A}\mathcal{A}} + \nu_{\mathcal{R}\mathcal{R}})}}_{T_1}, \underbrace{\frac{1}{\min\{\nu_{\mathcal{A}\mathcal{A}}, \nu_{\mathcal{R}\mathcal{R}}\}}}_{T_2}, \underbrace{\frac{2 - I_2}{2\tau_{\mathcal{A}\mathcal{R}}}}_{T_3} \right\},$$

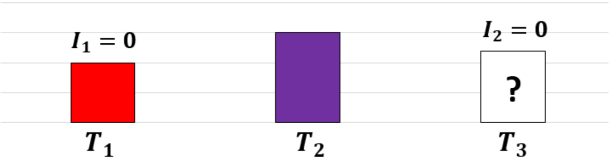
because $\min\{\nu_{\mathcal{A}\mathcal{A}}, \nu_{\mathcal{R}\mathcal{R}}\} \leq \frac{\nu_{\mathcal{A}\mathcal{A}} + \nu_{\mathcal{R}\mathcal{R}}}{2} \leq \frac{\nu_{\mathcal{A}\mathcal{A}} + \nu_{\mathcal{R}\mathcal{R}}}{2 - I_1}$.



- Achievability:

$$p > \frac{\log n}{n} \cdot \max\left\{ \underbrace{\frac{(2 - I_1)}{(\nu_{\mathcal{AA}} + \nu_{\mathcal{RR}})}}_{T_1}, \underbrace{\frac{1}{\min\{\nu_{\mathcal{AA}}, \nu_{\mathcal{RR}}\}}}_{T_2}, \underbrace{\frac{2 - I_2}{2\tau_{\mathcal{AR}}}}_{T_3} \right\},$$

because $\min\{\nu_{\mathcal{AA}}, \nu_{\mathcal{RR}}\} \leq \frac{\nu_{\mathcal{AA}} + \nu_{\mathcal{RR}}}{2} \leq \frac{\nu_{\mathcal{AA}} + \nu_{\mathcal{RR}}}{2 - I_1}$.



- $T_1 \leq T_2 \Rightarrow$ Row graph G_1 does not help

Example 1 when $n = m$ (also includes $n \leq m$)

- Achievability:

$$p > \frac{\log n}{n} \cdot \max\left\{ \underbrace{\frac{(2 - I_1)}{(\nu_{\mathcal{AA}} + \nu_{\mathcal{RR}})}}_{T_1}, \underbrace{\frac{1}{\min\{\nu_{\mathcal{AA}}, \nu_{\mathcal{RR}}\}}}_{T_2}, \underbrace{\frac{2 - I_2}{2\tau_{\mathcal{AR}}}}_{T_3} \right\},$$

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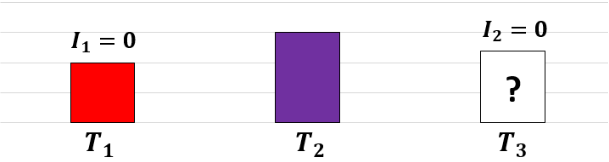
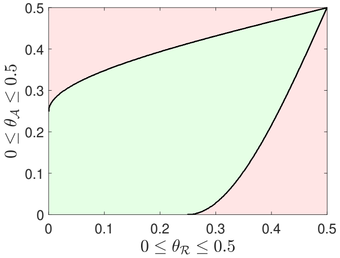
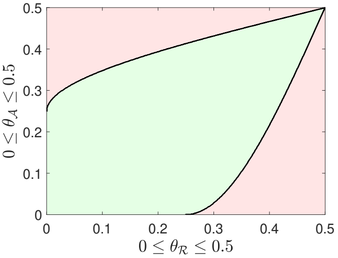


- $T_1 \leq T_2 \Rightarrow$ Row graph G_1 does not help
- Does column graph G_2 help?

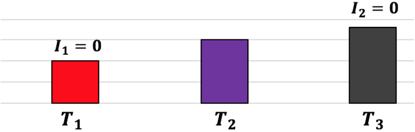
Example 1 when $n = m$ (also includes $n \leq m$)



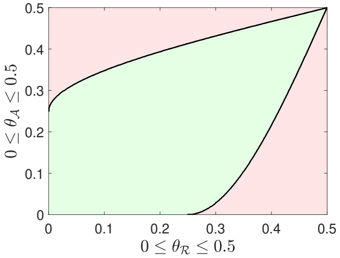


$$\max\left\{ \underbrace{\frac{(2 - I_1)}{(\nu_{\mathcal{A}\mathcal{A}} + \nu_{\mathcal{R}\mathcal{R}})}}_{T_1}, \underbrace{\frac{1}{\min\{\nu_{\mathcal{A}\mathcal{A}}, \nu_{\mathcal{R}\mathcal{R}}\}}}_{T_2}, \underbrace{\frac{2 - I_2}{2\tau_{\mathcal{A}\mathcal{R}}}}_{T_3} \right\}$$

(θ_A, θ_R) in green region

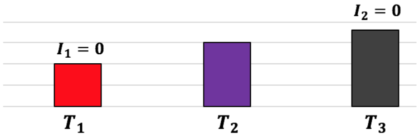


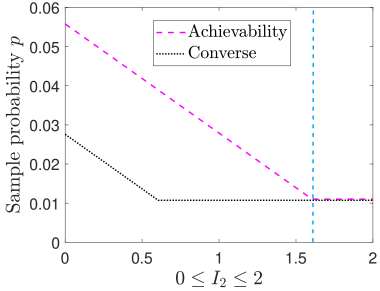
Example 1 when $n = m$ (also includes $n \leq m$)

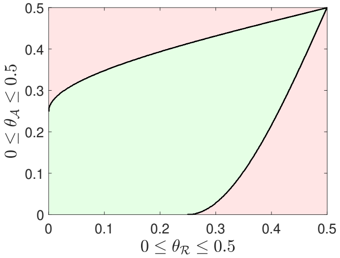


$$\max\left\{ \underbrace{\frac{(2 - I_1)}{(\nu_{\mathcal{A}\mathcal{A}} + \nu_{\mathcal{R}\mathcal{R}})}}_{T_1}, \underbrace{\frac{1}{\min\{\nu_{\mathcal{A}\mathcal{A}}, \nu_{\mathcal{R}\mathcal{R}}\}}}_{T_2}, \underbrace{\frac{2 - I_2}{2\tau_{\mathcal{A}\mathcal{R}}}}_{T_3} \right\}$$

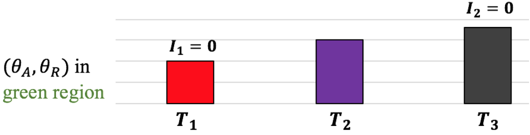
(θ_A, θ_R) in green region

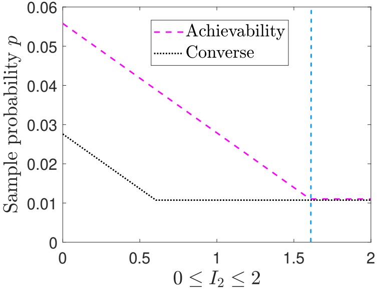




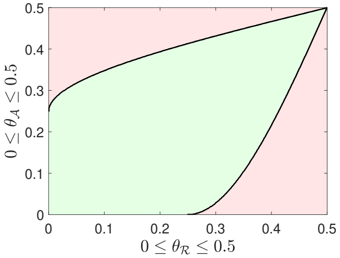


$$\max \left\{ \underbrace{\frac{(2 - I_1)}{(\nu_{\mathcal{AA}} + \nu_{\mathcal{RR}})}}_{T_1}, \underbrace{\frac{1}{\min\{\nu_{\mathcal{AA}}, \nu_{\mathcal{RR}}\}}}_{T_2}, \underbrace{\frac{2 - I_2}{2\tau_{\mathcal{AR}}}}_{T_3} \right\}$$



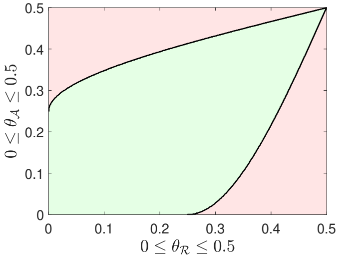


- Observing column graph G_2 reduces sample complexity nmp ☺
- As I_2 ↑, p first ↓, then stays constant.
- Intuition: $\theta_{\mathcal{A}} \approx \theta_{\mathcal{R}} \Rightarrow$ additional information (i.e., G_2) helps in distinguishing communities.



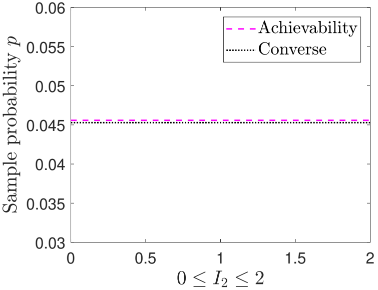
$$\max \Bigg\{ \underbrace{\frac{(2 - I_1)}{(\nu_{\mathcal{AA}} + \nu_{\mathcal{RR}})}}_{T_1}, \underbrace{\frac{1}{\min\{\nu_{\mathcal{AA}}, \nu_{\mathcal{RR}}\}}}_{T_2}, \underbrace{\frac{2 - I_2}{2\tau_{\mathcal{AR}}}}_{T_3} \Bigg\}$$

(θ_A, θ_R) in red region

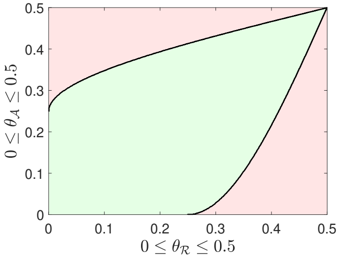


$$\max \Bigg\{ \underbrace{\frac{(2 - I_1)}{(\nu_{\mathcal{AA}} + \nu_{\mathcal{RR}})}}_{T_1}, \underbrace{\frac{1}{\min\{\nu_{\mathcal{AA}}, \nu_{\mathcal{RR}}\}}}_{T_2}, \underbrace{\frac{2 - I_2}{2\tau_{\mathcal{AR}}}}_{T_3} \Bigg\}$$

(θ_A, θ_R) in red region

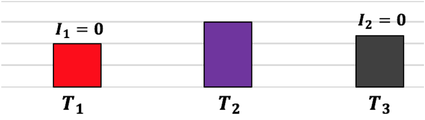


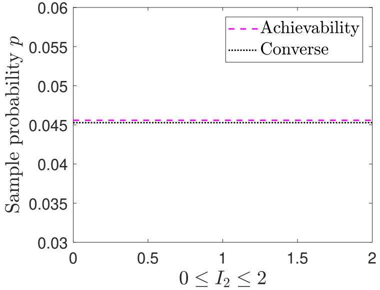
Example 1 when $n = m$ (also includes $n \le m$)



$$\max\left\{ \underbrace{\frac{(2 - I_1)}{(\nu_{\mathcal{AA}} + \nu_{\mathcal{RR}})}}_{T_1}, \underbrace{\frac{1}{\min\{\nu_{\mathcal{AA}}, \nu_{\mathcal{RR}}\}}}_{T_2}, \underbrace{\frac{2 - I_2}{2\tau_{\mathcal{AR}}}}_{T_3} \right\}$$

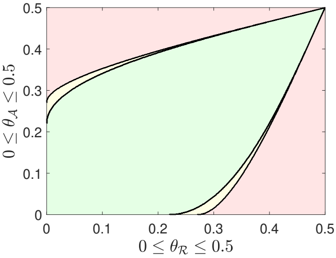
(θ_A, θ_R) in red region

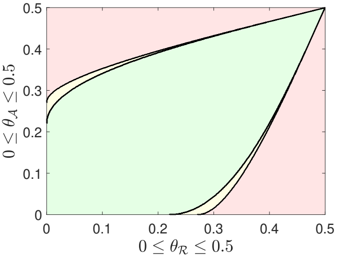




- Column graph G_2 does not help ☹
- Intuition: $\theta_{\mathcal{A}}$ is far from $\theta_{\mathcal{R}}$ $\Rightarrow$ Ample information to distinguish communities.

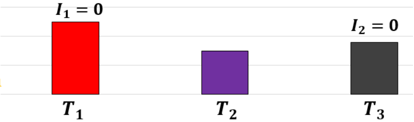
Example 2 ($n = 5m$)





$$\max\left\{ \underbrace{\frac{5(2-I_1)}{(\nu_{\mathcal{A}\mathcal{A}}+\nu_{\mathcal{R}\mathcal{R}})}}_{T_1}, \underbrace{\frac{1}{\min\{\nu_{\mathcal{A}\mathcal{A}},\nu_{\mathcal{R}\mathcal{R}}\}}}_{T_2}, \underbrace{\frac{2-I_2}{2\tau_{\mathcal{A}\mathcal{R}}}}_{T_3} \right\}$$

(θ_A, θ_R) in yellow region



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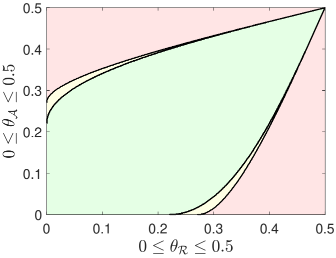
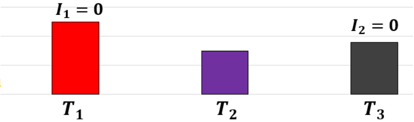
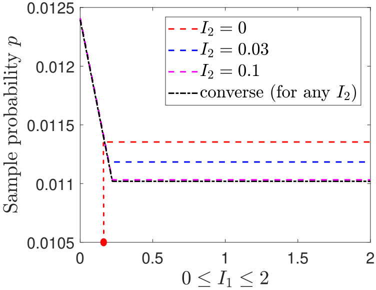
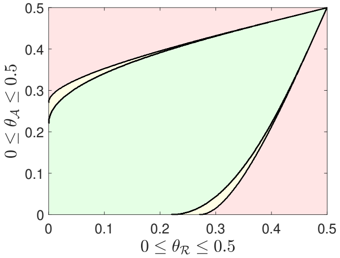


$$\max\left\{ \underbrace{\frac{5(2-I_1)}{(\nu_{\mathcal{AA}}+\nu_{\mathcal{RR}})}}_{T_1}, \underbrace{\frac{1}{\min\{\nu_{\mathcal{AA}},\nu_{\mathcal{RR}}\}}}_{T_2}, \underbrace{\frac{2-I_2}{2\tau_{\mathcal{AR}}}}_{T_3} \right\}$$

(θ_A, θ_R) in yellow region

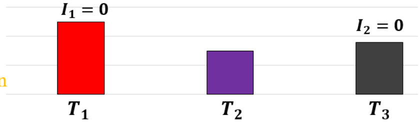


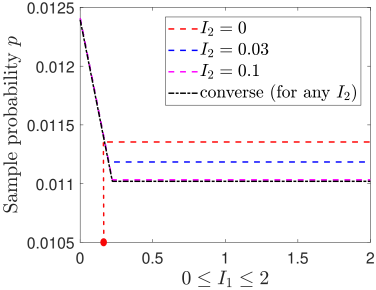




$$\max\left\{ \underbrace{\frac{5(2-I_1)}{(\nu_{\mathcal{AA}}+\nu_{\mathcal{RR}})}}_{T_1}, \underbrace{\frac{1}{\min\{\nu_{\mathcal{AA}},\nu_{\mathcal{RR}}\}}}_{T_2}, \underbrace{\frac{2-I_2}{2\tau_{\mathcal{AR}}}}_{T_3} \right\}$$

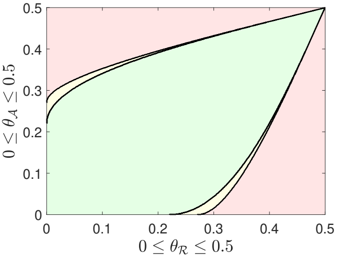
(θ_A, θ_R) in yellow region





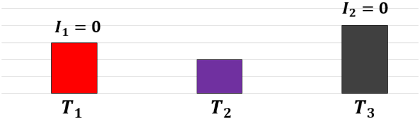
- Observing row graph G_1 reduces sample complexity;
- Column graph G_2 also helps if I_1 exceeds red point.

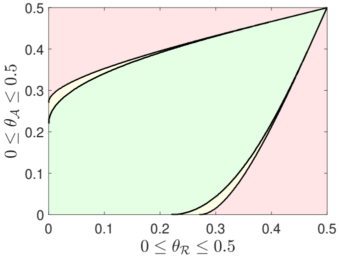
Example 2 ($n = 5m$)



$$\max\left\{ \underbrace{\frac{5(2-I_1)}{(\nu_{\mathcal{A}\mathcal{A}}+\nu_{\mathcal{R}\mathcal{R}})}}_{T_1}, \underbrace{\frac{1}{\min\{\nu_{\mathcal{A}\mathcal{A}}, \nu_{\mathcal{R}\mathcal{R}}\}}}_{T_2}, \underbrace{\frac{2-I_2}{2\tau_{\mathcal{A}\mathcal{R}}}}_{T_3} \right\}$$

(θ_A, θ_R) in green region





$$\max\left\{ \underbrace{\frac{5(2-I_1)}{(\nu_{\mathcal{AA}}+\nu_{\mathcal{RR}})}}_{T_1}, \underbrace{\frac{1}{\min\{\nu_{\mathcal{AA}},\nu_{\mathcal{RR}}\}}}_{T_2}, \underbrace{\frac{2-I_2}{2\tau_{\mathcal{AR}}}}_{T_3} \right\}$$

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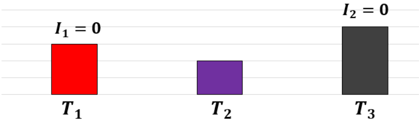
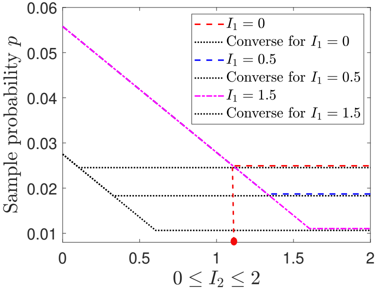
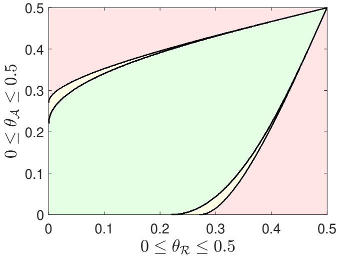
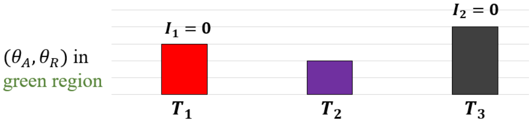
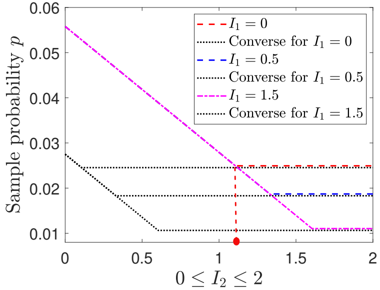






$$\max\left\{ \underbrace{\frac{5(2-I_1)}{(\nu_{\mathcal{AA}}+\nu_{\mathcal{RR}})}}_{T_1}, \underbrace{\frac{1}{\min\{\nu_{\mathcal{AA}},\nu_{\mathcal{RR}}\}}}_{T_2}, \underbrace{\frac{2-I_2}{2\tau_{\mathcal{AR}}}}_{T_3} \right\}$$





- Observing column graph G_2 reduces sample complexity;
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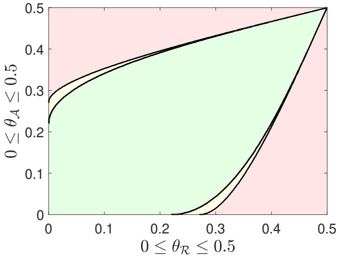
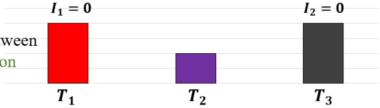
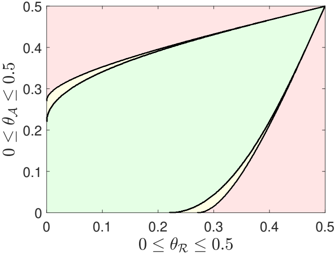


$$\max \left\{ \underbrace{\frac{5(2 - I_1)}{(\nu_{\mathcal{A}\mathcal{A}} + \nu_{\mathcal{R}\mathcal{R}})}}_{T_1}, \underbrace{\frac{1}{\min\{\nu_{\mathcal{A}\mathcal{A}}, \nu_{\mathcal{R}\mathcal{R}}\}}}_{T_2}, \underbrace{\frac{2 - I_2}{2\tau_{\mathcal{A}\mathcal{R}}}}_{T_3} \right\}$$

(θ_A, θ_R) boundary between yellow and green region

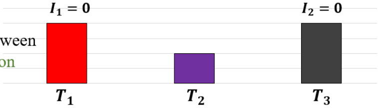


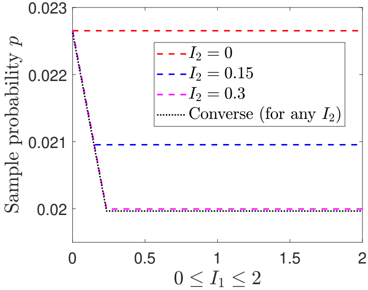
Example 2 ($n = 5m$)



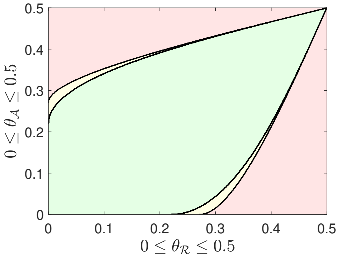
$$\max\left\{ \underbrace{\frac{5(2-I_1)}{(\nu_{\mathcal{AA}}+\nu_{\mathcal{RR}})}}_{T_1}, \underbrace{\frac{1}{\min\{\nu_{\mathcal{AA}},\nu_{\mathcal{RR}}\}}}_{T_2}, \underbrace{\frac{2-I_2}{2\tau_{\mathcal{AR}}}}_{T_3} \right\}$$

(θ_A, θ_R) boundary between yellow and green region



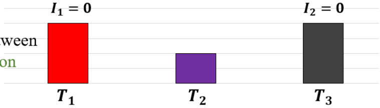


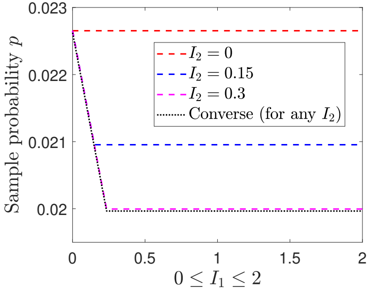
Example 2 ($n = 5m$)



$$\max \left\{ \underbrace{\frac{5(2 - I_1)}{(\nu_{\mathcal{A}\mathcal{A}} + \nu_{\mathcal{R}\mathcal{R}})}}_{T_1}, \underbrace{\frac{1}{\min\{\nu_{\mathcal{A}\mathcal{A}}, \nu_{\mathcal{R}\mathcal{R}}\}}}_{T_2}, \underbrace{\frac{2 - I_2}{2\tau_{\mathcal{A}\mathcal{R}}}}_{T_3} \right\}$$

(θ_A, θ_R) boundary between yellow and green region





- Observing both G_1 and G_2 reduces sample complexity;
- Observing only one graph is equivalent to observing neither;
- Synergistic effect of observing both graphs G_1 and G_2!

Proof Sketch of Achievability I

Maximum likelihood estimator ϕ_{ML}

- Negative log-likelihood of ξ:

$$L(\xi) \triangleq -\log \mathbb{P}_{\xi}(V^{\Omega}, G_1, G_2).$$

- Estimation rule:

$$\widehat{\xi} = \phi_{\mathrm{ML}}(V^{\Omega}, G_1, G_2) = \operatorname*{argmin}_{\xi \in \Xi} L(\xi).$$

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- Probability of error:

$$\begin{aligned} P_{\mathrm{err}}(\phi_{\mathrm{ML}}) &= \mathbb{P}_{\xi^*}\left(\phi_{\mathrm{ML}}(V^{\Omega}, G_1, G_2) \neq \xi^*\right) \\ &\leq \sum_{\xi \in \Xi \setminus \{\xi^*\}} \mathbb{P}_{\xi^*}\left(L(\xi) \leq L(\xi^*)\right). \end{aligned}$$

Proof Sketch of Achievability II

1. For each $\xi \neq \xi^*$, bound error probability $\mathbb{P}_{\xi^*}(L(\xi) \leq L(\xi^*))$

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   - Overlap between $\mathcal{M}$ and $\mathcal{W}$: $k_1 \triangleq |\xi_{\mathcal{M}} \setminus \xi^*_{\mathcal{M}}| = |\xi_{\mathcal{W}} \setminus \xi^*_{\mathcal{W}}|$

Proof Sketch of Achievability II

1. For each $\xi \neq \xi^*$, bound error probability $\mathbb{P}_{\xi^*}(L(\xi) \leq L(\xi^*))$
   - Overlap between $\mathcal{M}$ and $\mathcal{W}$: $k_1 \triangleq |\xi_{\mathcal{M}} \setminus \xi^*_{\mathcal{M}}| = |\xi_{\mathcal{W}} \setminus \xi^*_{\mathcal{W}}|$
   - Overlap between $\mathcal{A}$ and $\mathcal{R}$: $k_2 \triangleq |\xi_{\mathcal{A}} \setminus \xi^*_{\mathcal{A}}| = |\xi_{\mathcal{R}} \setminus \xi^*_{\mathcal{R}}|$

Proof Sketch of Achievability II

1. For each $\xi \neq \xi^*$, bound error probability $\mathbb{P}_{\xi^*}(L(\xi) \leq L(\xi^*))$
   - Overlap between $\mathcal{M}$ and $\mathcal{W}$: $k_1 \triangleq |\xi_{\mathcal{M}} \setminus \xi^*_{\mathcal{M}}| = |\xi_{\mathcal{W}} \setminus \xi^*_{\mathcal{W}}|$
   - Overlap between $\mathcal{A}$ and $\mathcal{R}$: $k_2 \triangleq |\xi_{\mathcal{A}} \setminus \xi^*_{\mathcal{A}}| = |\xi_{\mathcal{R}} \setminus \xi^*_{\mathcal{R}}|$
   - Number of distinct entries in B_ξ and B_{ξ^*}: $t_{\mathcal{A}\mathcal{A}}, t_{\mathcal{R}\mathcal{R}}$.

Proof Sketch of Achievability II

1. For each $\xi \neq \xi^*$, bound error probability $\mathbb{P}_{\xi^*}(L(\xi) \leq L(\xi^*))$
   - Overlap between $\mathcal{M}$ and $\mathcal{W}$: $k_1 \triangleq |\xi_{\mathcal{M}} \setminus \xi^*_{\mathcal{M}}| = |\xi_{\mathcal{W}} \setminus \xi^*_{\mathcal{W}}|$
   - Overlap between $\mathcal{A}$ and $\mathcal{R}$: $k_2 \triangleq |\xi_{\mathcal{A}} \setminus \xi^*_{\mathcal{A}}| = |\xi_{\mathcal{R}} \setminus \xi^*_{\mathcal{R}}|$
   - Number of distinct entries in B_ξ and B_{ξ^*}: $t_{\mathcal{A}\mathcal{A}}, t_{\mathcal{R}\mathcal{R}}$.
2. $\mathbb{P}_{\xi^*}(L(\xi) \leq L(\xi^*))$: function only of $(k_1, k_2, t_{\mathcal{A}\mathcal{A}}, t_{\mathcal{R}\mathcal{R}})$.

1. For each $\xi \neq \xi^*$, bound error probability $\mathbb{P}_{\xi^*}(L(\xi) \leq L(\xi^*))$
   - Overlap between $\mathcal{M}$ and $\mathcal{W}$: $k_1 \triangleq |\xi_{\mathcal{M}} \setminus \xi^*_{\mathcal{M}}| = |\xi_{\mathcal{W}} \setminus \xi^*_{\mathcal{W}}|$
   - Overlap between $\mathcal{A}$ and $\mathcal{R}$: $k_2 \triangleq |\xi_{\mathcal{A}} \setminus \xi^*_{\mathcal{A}}| = |\xi_{\mathcal{R}} \setminus \xi^*_{\mathcal{R}}|$
   - Number of distinct entries in B_ξ and B_{ξ^*}: $t_{\mathcal{A}\mathcal{A}}, t_{\mathcal{R}\mathcal{R}}$.
2. $\mathbb{P}_{\xi^*}(L(\xi) \leq L(\xi^*))$: function only of $(k_1, k_2, t_{\mathcal{A}\mathcal{A}}, t_{\mathcal{R}\mathcal{R}})$.
3. Partition $\xi \in \Xi \setminus \{\xi^*\}$ into different **types** $(k_1, k_2, t_{\mathcal{A}\mathcal{A}}, t_{\mathcal{R}\mathcal{R}})$

Proof Sketch of Achievability II

1. For each $\xi \neq \xi^*$, bound error probability $\mathbb{P}_{\xi^*}(L(\xi) \leq L(\xi^*))$
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   - Overlap between $\mathcal{A}$ and $\mathcal{R}$: $k_2 \triangleq |\xi_{\mathcal{A}} \setminus \xi^*_{\mathcal{A}}| = |\xi_{\mathcal{R}} \setminus \xi^*_{\mathcal{R}}|$
   - Number of distinct entries in B_ξ and B_{ξ^*}: $t_{\mathcal{A}\mathcal{A}}, t_{\mathcal{R}\mathcal{R}}$.
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3. Partition $\xi \in \Xi \setminus \{\xi^*\}$ into different **types** $(k_1, k_2, t_{\mathcal{A}\mathcal{A}}, t_{\mathcal{R}\mathcal{R}})$
4. For each type $(k_1, k_2, t_{\mathcal{A}\mathcal{A}}, t_{\mathcal{R}\mathcal{R}})$

Proof Sketch of Achievability II

1. For each $\xi \neq \xi^*$, bound error probability $\mathbb{P}_{\xi^*}(L(\xi) \leq L(\xi^*))$
   - Overlap between $\mathcal{M}$ and $\mathcal{W}$: $k_1 \triangleq |\xi_{\mathcal{M}} \setminus \xi^*_{\mathcal{M}}| = |\xi_{\mathcal{W}} \setminus \xi^*_{\mathcal{W}}|$
   - Overlap between $\mathcal{A}$ and $\mathcal{R}$: $k_2 \triangleq |\xi_{\mathcal{A}} \setminus \xi^*_{\mathcal{A}}| = |\xi_{\mathcal{R}} \setminus \xi^*_{\mathcal{R}}|$
   - Number of distinct entries in B_ξ and B_{ξ^*}: $t_{\mathcal{A}\mathcal{A}}, t_{\mathcal{R}\mathcal{R}}$.
2. $\mathbb{P}_{\xi^*}(L(\xi) \leq L(\xi^*))$: function only of $(k_1, k_2, t_{\mathcal{A}\mathcal{A}}, t_{\mathcal{R}\mathcal{R}})$.
3. Partition $\xi \in \Xi \setminus \{\xi^*\}$ into different **types** $(k_1, k_2, t_{\mathcal{A}\mathcal{A}}, t_{\mathcal{R}\mathcal{R}})$
4. For each type $(k_1, k_2, t_{\mathcal{A}\mathcal{A}}, t_{\mathcal{R}\mathcal{R}})$
   - Calculate $\mathbb{P}_{\xi^*}(L(\xi) \leq L(\xi^*))$ by applying the Chernoff bound

$$\mathbb{P}(X > \xi) \leq \min_{t>0} e^{-t\xi} \cdot \mathbb{E}(e^{tX}), \text{ with } t = \frac{1}{2}$$

1. For each $\xi \neq \xi^*$, bound error probability $\mathbb{P}_{\xi^*}(L(\xi) \leq L(\xi^*))$
   - Overlap between $\mathcal{M}$ and $\mathcal{W}$: $k_1 \triangleq |\xi_{\mathcal{M}} \setminus \xi^*_{\mathcal{M}}| = |\xi_{\mathcal{W}} \setminus \xi^*_{\mathcal{W}}|$
   - Overlap between $\mathcal{A}$ and $\mathcal{R}$: $k_2 \triangleq |\xi_{\mathcal{A}} \setminus \xi^*_{\mathcal{A}}| = |\xi_{\mathcal{R}} \setminus \xi^*_{\mathcal{R}}|$
   - Number of distinct entries in B_ξ and B_{ξ^*}: $t_{\mathcal{A}\mathcal{A}}, t_{\mathcal{R}\mathcal{R}}$.
2. $\mathbb{P}_{\xi^*}(L(\xi) \leq L(\xi^*))$: function only of $(k_1, k_2, t_{\mathcal{A}\mathcal{A}}, t_{\mathcal{R}\mathcal{R}})$.
3. Partition $\xi \in \Xi \setminus \{\xi^*\}$ into different **types** $(k_1, k_2, t_{\mathcal{A}\mathcal{A}}, t_{\mathcal{R}\mathcal{R}})$
4. For each type $(k_1, k_2, t_{\mathcal{A}\mathcal{A}}, t_{\mathcal{R}\mathcal{R}})$
   - Calculate $\mathbb{P}_{\xi^*}(L(\xi) \leq L(\xi^*))$ by applying the Chernoff bound

$$\mathbb{P}(X > \xi) \leq \min_{t>0} e^{-t\xi} \cdot \mathbb{E}(e^{tX}), \text{ with } t = \frac{1}{2}$$

   - Count the number of elements in each **type classes**

Proof Sketch of Converse I

1. Maximum likelihood estimator ϕ_{ML} minimizes P_{err}

$$\inf_{\phi} P_{\mathrm{err}}(\phi) \geq \mathbb{P}_{\xi^*}\left(\phi_{\mathrm{ML}}(V^{\Omega}, G_1, G_2) \neq \xi^*\right).$$

Proof Sketch of Converse I

1. Maximum likelihood estimator ϕ_{ML} minimizes P_{err}

$$\inf_{\phi} P_{\mathrm{err}}(\phi) \geq \mathbb{P}_{\xi^*}\left(\phi_{\mathrm{ML}}(V^{\Omega}, G_1, G_2) \neq \xi^*\right).$$

2. Restrict analysis to a subset of Ξ (likeliest to cause errors)

Proof Sketch of Converse I

1. Maximum likelihood estimator ϕ_{ML} minimizes P_{err}

$$\inf_{\phi} P_{\mathrm{err}}(\phi) \geq \mathbb{P}_{\xi^*}\left(\phi_{\mathrm{ML}}(V^{\Omega}, G_1, G_2) \neq \xi^*\right).$$

2. Restrict analysis to a subset of Ξ (likeliest to cause errors)
    - $(k_1, k_2, t_{\mathcal{A}\mathcal{A}}, t_{\mathcal{R}\mathcal{R}}) = (0, 0, 1, 0)$: $P_{\mathrm{err}} \to 1$ if $p < \frac{\log m}{\nu_{\mathcal{A}\mathcal{A}} n}$

Proof Sketch of Converse I

1. Maximum likelihood estimator ϕ_{ML} minimizes P_{err}

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   - $(k_1, k_2, t_{\mathcal{A}\mathcal{A}}, t_{\mathcal{R}\mathcal{R}}) = (0, 0, 0, 1)$: $P_{\mathrm{err}} \to 1$ if $p < \frac{\log m}{\nu_{\mathcal{R}\mathcal{R}} n}$

Proof Sketch of Converse I

1. Maximum likelihood estimator ϕ_{ML} minimizes P_{err}

$$\inf_{\phi} P_{\mathrm{err}}(\phi) \geq \mathbb{P}_{\xi^*}\left(\phi_{\mathrm{ML}}(V^{\Omega}, G_1, G_2) \neq \xi^*\right).$$

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   - $(k_1, k_2, t_{\mathcal{A}\mathcal{A}}, t_{\mathcal{R}\mathcal{R}}) = (0, 0, 1, 0)$: $P_{\mathrm{err}} \to 1$ if $p < \frac{\log m}{\nu_{\mathcal{A}\mathcal{A}} n}$
   - $(k_1, k_2, t_{\mathcal{A}\mathcal{A}}, t_{\mathcal{R}\mathcal{R}}) = (0, 0, 0, 1)$: $P_{\mathrm{err}} \to 1$ if $p < \frac{\log m}{\nu_{\mathcal{R}\mathcal{R}} n}$
   - $(k_1, k_2, t_{\mathcal{A}\mathcal{A}}, t_{\mathcal{R}\mathcal{R}}) = (1, 0, 0, 0)$: $P_{\mathrm{err}} \to 1$ if $p < \frac{(2-I_1)\log n}{(\nu_{\mathcal{A}\mathcal{A}}+\nu_{\mathcal{R}\mathcal{R}})m}$.

1. Maximum likelihood estimator ϕ_{ML} minimizes P_{err}

$$\inf_{\phi} P_{\mathrm{err}}(\phi) \geq \mathbb{P}_{\xi^*}\left(\phi_{\mathrm{ML}}(V^{\Omega}, G_1, G_2) \neq \xi^*\right).$$

2. Restrict analysis to a subset of Ξ (likeliest to cause errors)
   - $(k_1, k_2, t_{\mathcal{A}\mathcal{A}}, t_{\mathcal{R}\mathcal{R}}) = (0, 0, 1, 0)$: $P_{\mathrm{err}} \to 1$ if $p < \frac{\log m}{\nu_{\mathcal{A}\mathcal{A}} n}$
   - $(k_1, k_2, t_{\mathcal{A}\mathcal{A}}, t_{\mathcal{R}\mathcal{R}}) = (0, 0, 0, 1)$: $P_{\mathrm{err}} \to 1$ if $p < \frac{\log m}{\nu_{\mathcal{R}\mathcal{R}} n}$
   - $(k_1, k_2, t_{\mathcal{A}\mathcal{A}}, t_{\mathcal{R}\mathcal{R}}) = (1, 0, 0, 0)$: $P_{\mathrm{err}} \to 1$ if $p < \frac{(2-I_1)\log n}{(\nu_{\mathcal{A}\mathcal{A}}+\nu_{\mathcal{R}\mathcal{R}})m}$.
   - $(k_1, k_2, t_{\mathcal{A}\mathcal{A}}, t_{\mathcal{R}\mathcal{R}}) = (0, 1, 0, 0)$: $P_{\mathrm{err}} \to 1$ if $p < \frac{(1-I_2)\log m}{2\tau_{\mathcal{A}\mathcal{R}} n}$

Proof Sketch of Converse I

1. Maximum likelihood estimator ϕ_{ML} minimizes P_{err}

$$\inf_{\phi} P_{\mathrm{err}}(\phi) \geq \mathbb{P}_{\xi^*}\left(\phi_{\mathrm{ML}}(V^{\Omega}, G_1, G_2) \neq \xi^*\right).$$

2. Restrict analysis to a subset of Ξ (likeliest to cause errors)
   - $(k_1, k_2, t_{\mathcal{A}\mathcal{A}}, t_{\mathcal{R}\mathcal{R}}) = (0,0,1,0)$: $P_{\mathrm{err}} \to 1$ if $p < \frac{\log m}{\nu_{\mathcal{A}\mathcal{A}} n}$
   - $(k_1, k_2, t_{\mathcal{A}\mathcal{A}}, t_{\mathcal{R}\mathcal{R}}) = (0,0,0,1)$: $P_{\mathrm{err}} \to 1$ if $p < \frac{\log m}{\nu_{\mathcal{R}\mathcal{R}} n}$
   - $(k_1, k_2, t_{\mathcal{A}\mathcal{A}}, t_{\mathcal{R}\mathcal{R}}) = (1,0,0,0)$: $P_{\mathrm{err}} \to 1$ if $p < \frac{(2-I_1)\log n}{(\nu_{\mathcal{A}\mathcal{A}}+\nu_{\mathcal{R}\mathcal{R}})m}$.
   - $(k_1, k_2, t_{\mathcal{A}\mathcal{A}}, t_{\mathcal{R}\mathcal{R}}) = (0,1,0,0)$: $P_{\mathrm{err}} \to 1$ if $p < \frac{(1-I_2)\log m}{2\tau_{\mathcal{A}\mathcal{R}} n}$
   - Tightness of Chernoff bound

1. Maximum likelihood estimator ϕ_{ML} minimizes P_{err}

$$\inf_{\phi} P_{\mathrm{err}}(\phi) \geq \mathbb{P}_{\xi^*}\left(\phi_{\mathrm{ML}}(V^{\Omega}, G_1, G_2) \neq \xi^*\right).$$

2. Restrict analysis to a subset of Ξ (likeliest to cause errors)
   - $(k_1, k_2, t_{\mathcal{A}\mathcal{A}}, t_{\mathcal{R}\mathcal{R}}) = (0,0,1,0)$: $P_{\mathrm{err}} \to 1$ if $p < \frac{\log m}{\nu_{\mathcal{A}\mathcal{A}} n}$
   - $(k_1, k_2, t_{\mathcal{A}\mathcal{A}}, t_{\mathcal{R}\mathcal{R}}) = (0,0,0,1)$: $P_{\mathrm{err}} \to 1$ if $p < \frac{\log m}{\nu_{\mathcal{R}\mathcal{R}} n}$
   - $(k_1, k_2, t_{\mathcal{A}\mathcal{A}}, t_{\mathcal{R}\mathcal{R}}) = (1,0,0,0)$: $P_{\mathrm{err}} \to 1$ if $p < \frac{(2-I_1)\log n}{(\nu_{\mathcal{A}\mathcal{A}}+\nu_{\mathcal{R}\mathcal{R}})m}$.
   - $(k_1, k_2, t_{\mathcal{A}\mathcal{A}}, t_{\mathcal{R}\mathcal{R}}) = (0,1,0,0)$: $P_{\mathrm{err}} \to 1$ if $p < \frac{(1-I_2)\log m}{2\tau_{\mathcal{A}\mathcal{R}} n}$
   - Tightness of Chernoff bound

3. Prob. of error tends to 1 if

$$p < \max\left\{\frac{(2-I_1)\log n}{(\nu_{\mathcal{A}\mathcal{A}}+\nu_{\mathcal{R}\mathcal{R}})m}, \frac{\log m}{\min\{\nu_{\mathcal{A}\mathcal{A}}, \nu_{\mathcal{R}\mathcal{R}}\}\cdot n}, \frac{(1-I_2)\log m}{2\tau_{\mathcal{A}\mathcal{R}} n}\right\}.$$

Proof Sketch of Converse II

1. Maximum likelihood estimator ϕ_{ML} minimizes P_{err}

$$\inf_{\phi} P_{\mathrm{err}}(\phi) \geq \mathbb{P}_{\xi^*}\left(\phi_{\mathrm{ML}}(V^{\Omega}, G_1, G_2) \neq \xi^*\right).$$

2. Restrict analysis to a subset of Ξ (likeliest to cause errors)
   - $(k_1, k_2, t_{\mathcal{AA}}, t_{\mathcal{RR}}) = (0,0,1,0)$: $P_{\mathrm{err}} \to 1$ if $p < \frac{\log m}{\nu_{\mathcal{AA}} n}$
   - $(k_1, k_2, t_{\mathcal{AA}}, t_{\mathcal{RR}}) = (0,0,0,1)$: $P_{\mathrm{err}} \to 1$ if $p < \frac{\log m}{\nu_{\mathcal{RR}} n}$
   - $(k_1, k_2, t_{\mathcal{AA}}, t_{\mathcal{RR}}) = (1,0,0,0)$: $P_{\mathrm{err}} \to 1$ if $p < \frac{(2-I_1)\log n}{(\nu_{\mathcal{AA}}+\nu_{\mathcal{RR}})m}$.
   - $(k_1, k_2, t_{\mathcal{AA}}, t_{\mathcal{RR}}) = (0,1,0,0)$: $P_{\mathrm{err}} \to 1$ if $p < \frac{(1-I_2)\log m}{2\tau_{\mathcal{AR}} n}$
   - Tightness of Chernoff bound

3. Prob. of error tends to 1 if

$$p < \max\left\{\frac{(2-I_1)\log n}{(\nu_{\mathcal{AA}}+\nu_{\mathcal{RR}})m}, \frac{\log m}{\min\{\nu_{\mathcal{AA}}, \nu_{\mathcal{RR}}\}\cdot n}, \frac{(1-I_2)\log m}{2\tau_{\mathcal{AR}} n}\right\}.$$

Proof Sketch of Converse III

$$P_{\mathrm{suc}} \le \mathbb{P}_{\xi^*}\left(\bigcap_{\xi \in \Xi_{\xi^*}(1,0,0,0)} \{L(\xi) > L(\xi^*)\} \right)$$

$$= \mathbb{P}_{\xi^*}\left(\bigcap_{i \in [1:\frac{n}{2}], j \in [\frac{n}{2}+1:n]} \left\{ L\left(\xi_{\mathrm{row}}^{*(i,j)}\right) > L(\xi^*) \right\} \right)$$

Proof Sketch of Converse III

$$P_{\text{suc}} \le \mathbb{P}_{\xi^*}\left(\bigcap_{\xi \in \Xi_{\xi^*}(1,0,0,0)} \{L(\xi) > L(\xi^*)\} \right)$$

$$= \mathbb{P}_{\xi^*}\left(\bigcap_{i\in[1:\frac{n}{2}], j\in[\frac{n}{2}+1:n]} \left\{ L\left(\xi_{\text{row}}^{*(i,j)}\right) > L(\xi^*) \right\} \right)$$

- $\xi_{\text{row}}^{*(1,4)} \in \Xi_{\xi^*}(1,0,0,0)$: swap row 1 & row 4

ξ^*

	1	2	3	4	5	6
1	1	1	0	0	0	1
2	1	1	0	0	0	1
3	1	1	0	0	0	1
4	0	0	1	1	1	0
5	0	0	1	1	1	0
6	0	0	1	1	1	0

$\xi_{\text{row}}^{*(1,4)}$

	1	2	3	4	5	6
1	0	0	1	1	1	0
2	1	1	0	0	0	1
3	1	1	0	0	0	1
4	1	1	0	0	0	1
5	0	0	1	1	1	0
6	0	0	1	1	1	0

Proof Sketch of Converse III

$$P_{\text{suc}} \leq \mathbb{P}_{\xi^*}\left(\bigcap_{\xi \in \Xi_{\xi^*}(1,0,0,0)} \{L(\xi) > L(\xi^*)\} \right)$$

$$= \mathbb{P}_{\xi^*}\left(\bigcap_{i \in [1:\frac{n}{2}], j \in [\frac{n}{2}+1:n]} \left\{ L\left(\xi_{\text{row}}^{*(i,j)}\right) > L(\xi^*) \right\} \right)$$

- $\xi_{\text{row}}^{*(1,4)} \in \Xi_{\xi^*}(1,0,0,0)$: swap row 1 & row 4
- $\xi_{\text{row}}^{*(1,5)} \in \Xi_{\xi^*}(1,0,0,0)$: swap row 1 & row 5

ξ^*

	1	2	3	4	5	6
1	1	1	0	0	0	1
2	1	1	0	0	0	1
3	1	1	0	0	0	1
4	0	0	1	1	1	0
5	0	0	1	1	1	0
6	0	0	1	1	1	0

$\xi_{\text{row}}^{*(1,4)}$

	1	2	3	4	5	6
1	0	0	1	1	1	0
2	1	1	0	0	0	1
3	1	1	0	0	0	1
4	1	1	0	0	0	1
5	0	0	1	1	1	0
6	0	0	1	1	1	0

$\xi_{\text{row}}^{*(1,5)}$

	1	2	3	4	5	6
1	0	0	1	1	1	0
2	1	1	0	0	0	1
3	1	1	0	0	0	1
4	0	0	1	1	1	0
5	1	1	0	0	0	1
6	0	0	1	1	1	0

Proof Sketch of Converse III

$$P_{\mathrm{suc}} \leq \mathbb{P}_{\xi^*}\left(\bigcap_{\xi \in \Xi_{\xi^*}(1,0,0,0)} \{L(\xi) > L(\xi^*)\} \right)$$

$$= \mathbb{P}_{\xi^*}\left(\bigcap_{i \in [1:\frac{n}{2}], j \in [\frac{n}{2}+1:n]} \left\{ L\left(\xi_{\mathrm{row}}^{*(i,j)}\right) > L(\xi^*) \right\} \right)$$

- $\xi_{\mathrm{row}}^{*(1,4)} \in \Xi_{\xi^*}(1,0,0,0)$: swap row 1 & row 4
- $\xi_{\mathrm{row}}^{*(1,5)} \in \Xi_{\xi^*}(1,0,0,0)$: swap row 1 & row 5
- $\{L(\xi_{\mathrm{row}}^{*(1,4)}) > L(\xi^*)\}$ & $\{L(\xi_{\mathrm{row}}^{*(1,5)}) > L(\xi^*)\}$ correlated

ξ^*

	1	2	3	4	5	6
1	1	1	0	0	0	1
2	1	1	0	0	0	1
3	1	1	0	0	0	1
4	0	0	1	1	1	0
5	0	0	1	1	1	0
6	0	0	1	1	1	0

$\xi_{\mathrm{row}}^{*(1,4)}$

	1	2	3	4	5	6
1	0	0	1	1	1	0
2	1	1	0	0	0	1
3	1	1	0	0	0	1
4	1	1	0	0	0	1
5	0	0	1	1	1	0
6	0	0	1	1	1	0

$\xi_{\mathrm{row}}^{*(1,5)}$

	1	2	3	4	5	6
1	0	0	1	1	1	0
2	1	1	0	0	0	1
3	1	1	0	0	0	1
4	0	0	1	1	1	0
5	1	1	0	0	0	1
6	0	0	1	1	1	0

Proof Sketch of Converse III

$$P_{\mathrm{suc}} \le \mathbb{P}_{\xi^*}\left(\bigcap_{\xi \in \Xi_{\xi^*}(1,0,0,0)} \{L(\xi) > L(\xi^*)\} \right)$$

$$= \mathbb{P}_{\xi^*}\left(\bigcap_{i \in [1:\frac{n}{2}], j \in [\frac{n}{2}+1:n]} \left\{ L\left(\xi_{\mathrm{row}}^{*(i,j)}\right) > L(\xi^*) \right\} \right)$$

- $\xi_{\mathrm{row}}^{*(1,4)} \in \Xi_{\xi^*}(1,0,0,0)$: swap row 1 & row 4
- $\xi_{\mathrm{row}}^{*(1,5)} \in \Xi_{\xi^*}(1,0,0,0)$: swap row 1 & row 5
- $\{L(\xi_{\mathrm{row}}^{*(1,4)}) > L(\xi^*)\}$ & $\{L(\xi_{\mathrm{row}}^{*(1,5)}) > L(\xi^*)\}$ correlated
- Caused by row graph G_1 and row 1 of rating matrix

ξ^*

	1	2	3	4	5	6
1	1	1	0	0	0	1
2	1	1	0	0	0	1
3	1	1	0	0	0	1
4	0	0	1	1	1	0
5	0	0	1	1	1	0
6	0	0	1	1	1	0

$\xi_{\mathrm{row}}^{*(1,4)}$

	1	2	3	4	5	6
1	0	0	1	1	1	0
2	1	1	0	0	0	1
3	1	1	0	0	0	1
4	1	1	0	0	0	1
5	0	0	1	1	1	0
6	0	0	1	1	1	0

$\xi_{\mathrm{row}}^{*(1,5)}$

	1	2	3	4	5	6
1	0	0	1	1	1	0
2	1	1	0	0	0	1
3	1	1	0	0	0	1
4	0	0	1	1	1	0
5	1	1	0	0	0	1
6	0	0	1	1	1	0

Proof Sketch of Converse III

$$P_{\text{suc}} \le \mathbb{P}_{\xi^*}\left(\bigcap_{\xi \in \Xi_{\xi^*}(1,0,0,0)} \{L(\xi) > L(\xi^*)\} \right)$$

$$= \mathbb{P}_{\xi^*}\left(\bigcap_{i\in[1:\frac{n}{2}], j\in[\frac{n}{2}+1:n]} \left\{ L\left(\xi^{*(i,j)}_{\text{row}}\right) > L(\xi^*) \right\} \right)$$

- $\xi^{*(1,4)}_{\text{row}} \in \Xi_{\xi^*}(1,0,0,0)$: swap row 1 & row 4
- $\xi^{*(1,5)}_{\text{row}} \in \Xi_{\xi^*}(1,0,0,0)$: swap row 1 & row 5
- $\{L(\xi^{*(1,4)}_{\text{row}}) > L(\xi^*)\}$ & $\{L(\xi^{*(1,5)}_{\text{row}}) > L(\xi^*)\}$ correlated
- Caused by row graph G_1 and row 1 of rating matrix
- $\{L(\xi^{*(i,j)}_{\text{row}}) > L(\xi^*)\}_{i,j}$ are correlated

ξ^*

	1	2	3	4	5	6
1	1	1	0	0	0	1
2	1	1	0	0	0	1
3	1	1	0	0	0	1
4	0	0	1	1	1	0
5	0	0	1	1	1	0
6	0	0	1	1	1	0

$\xi^{*(1,4)}_{\text{row}}$

	1	2	3	4	5	6
1	0	0	1	1	1	0
2	1	1	0	0	0	1
3	1	1	0	0	0	1
4	1	1	0	0	0	1
5	0	0	1	1	1	0
6	0	0	1	1	1	0

$\xi^{*(1,5)}_{\text{row}}$

	1	2	3	4	5	6
1	0	0	1	1	1	0
2	1	1	0	0	0	1
3	1	1	0	0	0	1
4	0	0	1	1	1	0
5	1	1	0	0	0	1
6	0	0	1	1	1	0

Proof Sketch of Converse IV

- Extract

$$f(n) = \frac{n}{\log^2 n} \approx n$$

nodes that are not connected to one another. Exists w.h.p.

- Extract
$$\textcolor{red}{f(n)} = \frac{n}{\log^2 n} \approx n$$
nodes that are not connected to one another. Exists w.h.p.
- W.l.o.g., say $[1 : \textcolor{red}{f(n)}] \cup [\frac{n}{2} + 1 : \frac{n}{2} + \textcolor{red}{f(n)}]$

		1	2	3	4	5	6	7	8	9	10	11	12
$f(n)$	1		1			1					0		1
$f(n)$	2			1			0		0				1
$f(n)$	...	1			1			0			1		
	$\frac{n}{2}$		1			1				0		0	
$f(n)$	$\frac{n}{2}+1$				0						1		
$f(n)$	$\frac{n}{2}+2$	0				0			1			1	
$f(n)$	...		0		0				1	1			0
	n	0		0			1	1			1		

Proof Sketch of Converse IV

- Extract

$$f(n) = \frac{n}{\log^2 n} \approx n$$

nodes that are not connected to one another. Exists w.h.p.

- W.l.o.g., say $[1 : f(n)] \cup [\frac{n}{2} + 1 : \frac{n}{2} + f(n)]$

		1	2	3	4	5	6	7	8	9	10	11	12
$f(n)$	1		1			1					0		1
	2			1			0		0				1
	...	1			1			0			1		
	$\frac{n}{2}$		1			1				0		0	
$f(n)$	$\frac{n}{2}+1$				0						1		
	$\frac{n}{2}+2$	0				0			1			1	
	...		0		0				1	1			0
	n	0		0			1	1			1		

- Remove correlations caused by the row graph G_1

$$P_{\rm suc} \le \mathbb{P}_{\xi^*}\left(\bigcap_{i \in [1:f(n)], j \in [\frac{n}{2}+1:\frac{n}{2}+f(n)]} \left\{ L\left(\xi_{\rm row}^{*(i,j)}\right) > L(\xi^*) \right\} \right)$$

Proof Sketch of Converse V

- $\{L(\xi_{\text{row}}^{*(i,j)}) > L(\xi^*)\}_{i,j}$ are correlated (e.g., $\{L(\xi_{\text{row}}^{*(1,4)}) > L(\xi^*)\}$ & $\{L(\xi_{\text{row}}^{*(1,5)}) > L(\xi^*)\}$ correlated through row 1 of rating matrix)

Proof Sketch of Converse V

- $\{L(\xi_{\text{row}}^{*(i,j)}) > L(\xi^*)\}_{i,j}$ are correlated (e.g., $\{L(\xi_{\text{row}}^{*(1,4)}) > L(\xi^*)\}$ & $\{L(\xi_{\text{row}}^{*(1,5)}) > L(\xi^*)\}$ correlated through row 1 of rating matrix)

- Define new random matrices $\xi_{\text{row}}^{*(i)}, \xi_{\text{row}}^{*(j)}$

$\xi_{\text{row}}^{*(1)}$

	1	2	3	4	5	6
1	0	0	1	1	1	0
2	1	1	0	0	0	1
3	1	1	0	0	0	1
4	0	0	1	1	1	0
5	0	0	1	1	1	0
6	0	0	1	1	1	0

$\xi_{\text{row}}^{*(5)}$

	1	2	3	4	5	6
1	1	1	0	0	0	1
2	1	1	0	0	0	1
3	1	1	0	0	0	1
4	0	0	1	1	1	0
5	1	1	0	0	0	1
6	0	0	1	1	1	0

$$P_{\text{suc}} \le \mathbb{P}_{\xi^*}\left(\bigcap_{i\in[1:f(n)]}\left\{L(\xi_{\text{row}}^{*(i)}) > L(\xi^*)\right\}\right) + \mathbb{P}_{\xi^*}\left(\bigcap_{j\in[\frac{n}{2}+1:\frac{n}{2}+f(n)]}\left\{L(\xi_{\text{row}}^{*(j)}) > L(\xi^*)\right\}\right)$$

Proof Sketch of Converse V

- $\{L(\xi_{\text{row}}^{*(i,j)}) > L(\xi^*)\}_{i,j}$ are correlated (e.g., $\{L(\xi_{\text{row}}^{*(1,4)}) > L(\xi^*)\}$ & $\{L(\xi_{\text{row}}^{*(1,5)}) > L(\xi^*)\}$ correlated through row 1 of rating matrix)

- Define new random matrices $\xi_{\text{row}}^{*(i)}, \xi_{\text{row}}^{*(j)}$

$\xi_{\text{row}}^{*(1)}$

	1	2	3	4	5	6
1	0	0	1	1	1	0
2	1	1	0	0	0	1
3	1	1	0	0	0	1
4	0	0	1	1	1	0
5	0	0	1	1	1	0
6	0	0	1	1	1	0

$\xi_{\text{row}}^{*(5)}$

	1	2	3	4	5	6
1	1	1	0	0	0	1
2	1	1	0	0	0	1
3	1	1	0	0	0	1
4	0	0	1	1	1	0
5	1	1	0	0	0	1
6	0	0	1	1	1	0

$$P_{\text{suc}} \le \mathbb{P}_{\xi^*}\left(\bigcap_{i\in[1:f(n)]}\left\{L(\xi_{\text{row}}^{*(i)}) > L(\xi^*)\right\}\right) + \mathbb{P}_{\xi^*}\left(\bigcap_{j\in[\frac{n}{2}+1:\frac{n}{2}+f(n)]}\left\{L(\xi_{\text{row}}^{*(j)}) > L(\xi^*)\right\}\right)$$

- $\{L(\xi_{\text{row}}^{*(i)}) > L(\xi^*)\}_{i\in[1:f(n)]}$ are independent

Proof Sketch of Converse V

- $\{L(\xi_{\text{row}}^{*(i,j)}) > L(\xi^*)\}_{i,j}$ are correlated (e.g., $\{L(\xi_{\text{row}}^{*(1,4)}) > L(\xi^*)\}$ & $\{L(\xi_{\text{row}}^{*(1,5)}) > L(\xi^*)\}$ correlated through row 1 of rating matrix)

- Define new random matrices $\xi_{\text{row}}^{*(i)}, \xi_{\text{row}}^{*(j)}$

$\xi_{\text{row}}^{*(1)}$

	1	2	3	4	5	6
1	0	0	1	1	1	0
2	1	1	0	0	0	1
3	1	1	0	0	0	1
4	0	0	1	1	1	0
5	0	0	1	1	1	0
6	0	0	1	1	1	0

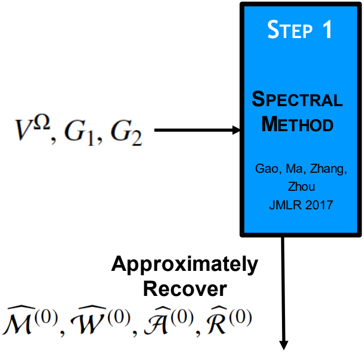
$\xi_{\text{row}}^{*(5)}$

	1	2	3	4	5	6
1	1	1	0	0	0	1
2	1	1	0	0	0	1
3	1	1	0	0	0	1
4	0	0	1	1	1	0
5	1	1	0	0	0	1
6	0	0	1	1	1	0

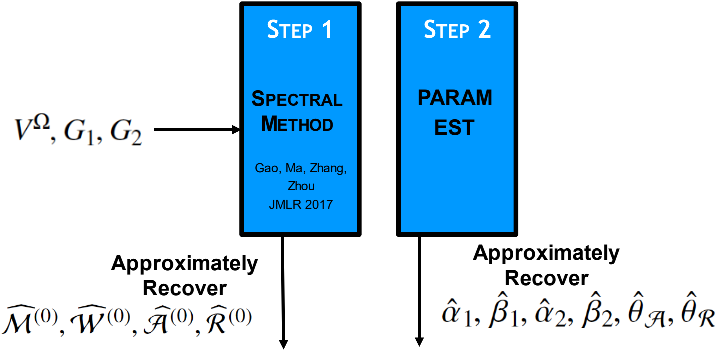
$$P_{\text{suc}} \leq \mathbb{P}_{\xi^*}\left(\bigcap_{i \in [1:f(n)]} \left\{ L(\xi_{\text{row}}^{*(i)}) > L(\xi^*) \right\} \right) + \mathbb{P}_{\xi^*}\left(\bigcap_{j \in [\frac{n}{2}+1:\frac{n}{2}+f(n)]} \left\{ L(\xi_{\text{row}}^{*(j)}) > L(\xi^*) \right\} \right)$$

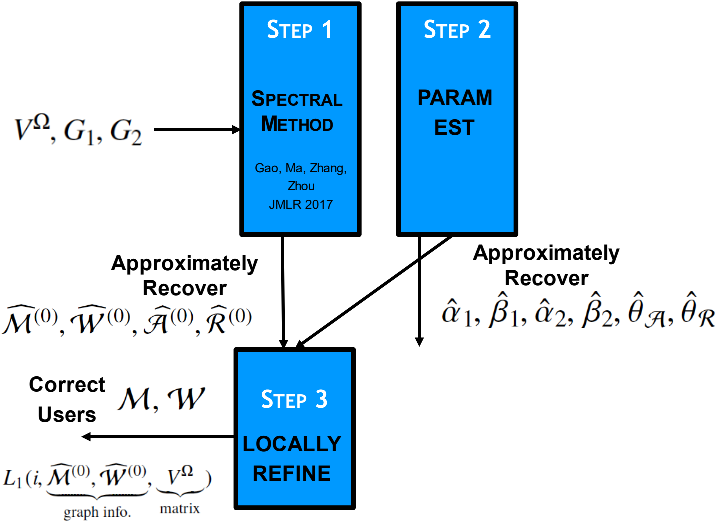
- $\{L(\xi_{\text{row}}^{*(i)}) > L(\xi^*)\}_{i \in [1:f(n)]}$ are independent
- $\{L(\xi_{\text{row}}^{*(j)}) > L(\xi^*)\}_{j \in [\frac{n}{2}+1:\frac{n}{2}+f(n)]}$ are independent

Efficient algorithm: Spectral + Local MLE

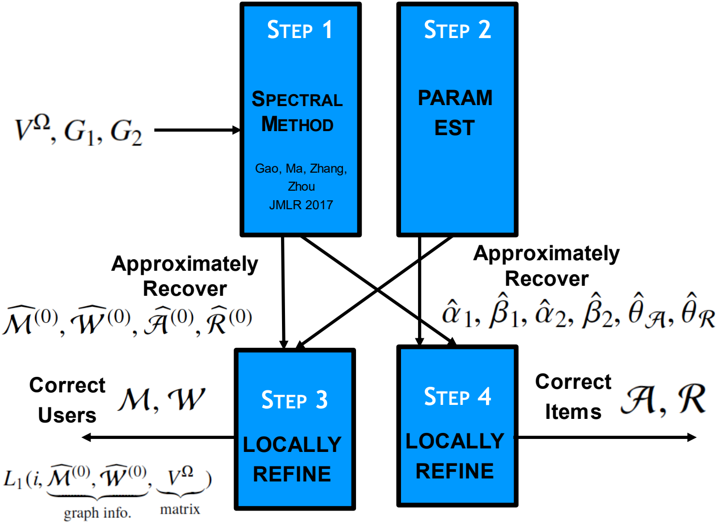


Efficient algorithm: Spectral + Local MLE



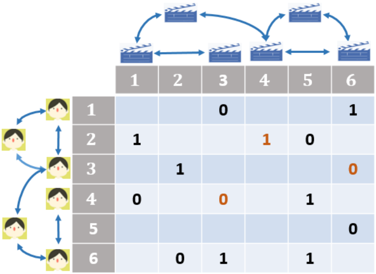

STEP 1
SPECTRAL METHOD
Gao, Ma, Zhang, Zhou
JMLR 2017
STEP 2
PARAM EST
V^Ω, G_1, G_2
Approximately Recover
$\widehat{\mathcal{M}}^{(0)}, \widehat{\mathcal{W}}^{(0)}, \widehat{\mathcal{A}}^{(0)}, \widehat{\mathcal{R}}^{(0)}$
Approximately Recover
$\hat{\alpha}_1, \hat{\beta}_1, \hat{\alpha}_2, \hat{\beta}_2, \hat{\theta}_{\mathcal{A}}, \hat{\theta}_{\mathcal{R}}$
STEP 3
LOCALLY REFINE
Correct Users
$\mathcal{M}, \mathcal{W}$
$L_1(i, \underbrace{\widehat{\mathcal{M}}^{(0)}, \widehat{\mathcal{W}}^{(0)}}_{\text{graph info.}}, \underbrace{V^\Omega}_{\text{matrix}})$

Efficient algorithm: Spectral + Local MLE



Conclusion

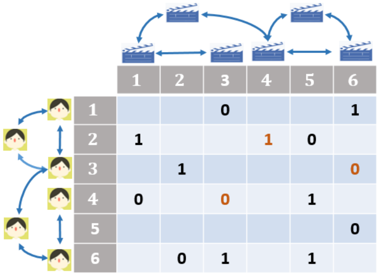
- Recommendation system with two-sided graph side-information



	1	2	3	4	5	6
1			0			1
2	1			1	0	
3		1				0
4	0		0		1	
5						0
6		0	1		1	

Conclusion

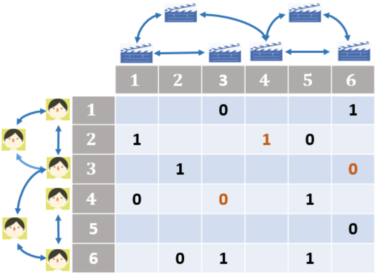
- Recommendation system with two-sided graph side-information



	1	2	3	4	5	6
1			0			1
2	1			1	0	
3		1				0
4	0		0		1	
5						0
6		0	1		1	

- Derived information-theoretic bounds on the sample complexity

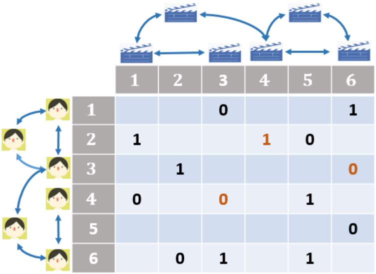
- Recommendation system with two-sided graph side-information



	1	2	3	4	5	6
1			0			1
2	1			1	0	
3		1				0
4	0		0		1	
5						0
6		0	1		1	

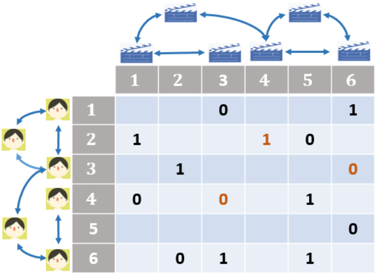
- Derived information-theoretic bounds on the sample complexity
- Synergistic effect of using both graphs

- Recommendation system with two-sided graph side-information



- Derived information-theoretic bounds on the sample complexity
- Synergistic effect of using both graphs
- Working on a spectral-based + successive refinement efficient algorithm attaining the information-theoretic limit

- Recommendation system with two-sided graph side-information



	1	2	3	4	5	6
1			0			1
2	1			1	0	
3		1				0
4	0		0		1	
5						0
6		0	1		1	

- Derived information-theoretic bounds on the sample complexity
- Synergistic effect of using both graphs
- Working on a spectral-based + successive refinement efficient algorithm attaining the information-theoretic limit
- https://arxiv.org/abs/1912.04099

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